ON ERGODICITY AND MIXING IN TOPOLOGICAL TRANSFORMATION GROUPS

By Harvey B. Keynes and James B. Robertson

1. Introduction. The measure-theoretic properties of ergodicity, weakly mixing, and strongly mixing have been extensively studied. In particular, there are the well-known proper value theorems for ergodicity and weakly mixing [4]. They essentially state that an invertible measure-preserving transformation is ergodic iff 1 is a simple proper value of the induced unitary operator on L_2, and weakly mixing iff 1 is the only proper value and it is simple.

In this paper, we study the corresponding topological properties in transformation groups and derive some analogous results. The notions of ergodicity and strongly mixing have been known for some time [3]; Furstenberg has recently introduced the notion of weakly mixing [1]. In §2, some consequences of these properties and the relationship between them are studied. As with the measure-theoretic notions, weakly mixing and strongly mixing are shown to be distinct notions. Furstenberg asked whether the product of a weakly mixing and ergodic transformation group is ergodic [1]. We provide a partial negative answer to this question.

Analogues of the proper value theorems for the case when a homeomorphism acts on a compact metric space are considered in §3. In the Banach algebra of bounded complex-valued functions which are continuous on a comeager set, the analogous result holds for ergodicity. It is shown that the result is not true in the space of continuous functions. Ergodicity of the powers of the homeomorphism are similarly characterized. Weakly mixing is shown to imply the appropriate statement in the above space. This condition is shown to be equivalent to the fact that every function whose orbit under the induced map is contained in a finite-dimensional space is necessarily a constant. We have not been able to show that these conditions are equivalent to weakly mixing; however, a partial converse is established.

Throughout this paper, **Z** will denote the additive group of integers.

2. General properties of ergodicity and mixing. Throughout this section, (X, T) will denote a transformation group, with T non-compact and X non-trivial.

Definition (2.1).

(1) (X, T) is *ergodic* if for every pair U, V of non-empty open sets in X, there is a $t \in T$ such that $Ut \cap V \neq \varnothing$. (In [1], this property is called transitive.)

(2) (X, T) is *weakly mixing* if the product transformation group $(X \times X, T)$ is ergodic.

Received June 23, 1967.