THE GENERALIZED HYPERGEOMETRIC DIFFERENTIAL EQUATION

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1. Introduction. Put

$$w = z^{\alpha}(\mu + z)\ {}_pF_{q-1}\left(\begin{matrix} a_p \\ 1 + b_{q-1}\end{matrix}\ \middle|\ \lambda(\nu + z)^{\gamma}\right), \tag{1.1}$$

where

$${}_pF_{q-1}\left(\begin{matrix} a_p \\ 1 + b_{q-1}\end{matrix}\ \middle|\ z\right) = \sum_{k=0}^{\infty} \frac{(a_p)_k}{(1 + b_{q-1})_k} \frac{z^k}{k!},$$

$$(a_p)_k = \prod_{r=1}^{p} (a_r)_k, \qquad (1 + b_{q-1})_k = \prod_{s=1}^{q-1} (1 + b_s)_k$$

and

$$(a)_k = a(a + 1) \cdots (a + k - 1), \qquad (a)_0 = 1.$$

The parameters α, β, μ, ν, λ, γ are arbitrary but $\lambda\gamma \neq 0$. Lavoie and Mongeau [1] have shown that w satisfies the differential equation

$$\phi_m H_m \psi_m = 0, \tag{1.2}$$

where $m = \max(p, q)$, ϕ_m is a row matrix:

$$\phi_m = (\sigma_{q-j}(b_{q-1}) - \lambda(\nu + z)^{\gamma}\sigma_{p-j}(a_p)) \qquad (j = 0, 1, \cdots, m), \tag{1.3}$$

$\sigma_j(a_p)$ denotes the j-th elementary symmetric function of $a_1, \cdots, a_p$ and similarly for $\sigma_j(b_{q-1})$; ψ_m is a column matrix with typical element

$$z^i \frac{d^i}{dz^i} \qquad (i = 0, 1, \cdots, m);$$

H_m is a square matrix of order $m + 1$ which is defined as the product of seven matrices each of which is described explicitly. Put

$$H_m = (h_{i,j}) \qquad (i, j = 0, 1, \cdots, m). \tag{1.4}$$

Then

$$h_{i,j} = 0 \qquad (0 \leq i < j \leq m), \tag{1.5}$$

so that H_m is a lower triangular matrix. The quantities $h_{i,j}$ are not determined explicitly. However it is shown that

$$h_{i,i} = (\gamma^{-1}N)^i \qquad (i = 0, 1, \cdots, m), \tag{1.6}$$

where $N = 1 + \nu z^{-1}$. Also $h_{i,j}$ is computed for $0 \leq j \leq i \leq 4$.

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