A GENERALIZED WEYL CRITERION

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Our object is to discuss the distribution of sequences of real numbers modulo one. Our results include Weyl's criterion on uniform distributions [3; 76] and several analogous results as special cases.

Let $\{x_i\}_1^\infty$ be an arbitrary sequence of real numbers and let β_i denote the fractional part of x_i.

For the sequence $\{\beta_i\}_1^\infty$ of fractional parts, we define the function F_n on $[0, 1]$ so that $F_n(x)$ is the fractional number of terms from the set $\{\beta_1, \cdots, \beta_n\}$ which lie in the interval $[0, x)$, that is, the number of such terms divided by n. Then for each $n > 0$, F_n is a nondecreasing, left-continuous function on $[0, 1]$ with $F(0) = 0$ and $F(1) = 1$.

Now, let $\phi_0, \phi_1, \phi_2, \cdots$ be an arbitrary sequence of complex-valued functions in $L_2[0, 1]$ which are closed with respect to the complex space $L_2[0, 1]$; i.e., if $f \in L_2[0, 1]$ and

$$\int_0^1 f(x)\phi_n(x)\,dx = 0 \quad \text{for all} \quad n \geq 0,$$

then $f(x) = 0$ almost everywhere on $[0, 1]$.

Next, we let g be a given real-valued function which is non-decreasing and continuous on $[0, 1]$ with $g(0) = 0$ and $g(1) = 1$. Further, let

$$a_k = \int_0^1 g(x)\phi_k(x)\,dx \quad \text{for} \quad k \geq 0$$

be the number obtained by taking the complex conjugate of the usual inner product of g and ϕ_k.

In terms of these various quantities, our theorem has the following form:

Theorem. *We have* $\lim_{n\to\infty} F_n(x) = g(x)$ *for all* $x \in [0, 1]$ *if and only if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \psi_\nu(\beta_j) = 0 \tag{1}$$

for each $\nu \geq 0$, *where the* ψ_ν *are defined for* $x \in [0, 1]$ *by*

$$\psi_\nu(x) = \begin{cases} 1 + \dfrac{1}{a_\nu} \displaystyle\int_1^x \phi_\nu(t)\,dt & \text{if} \quad a_\nu \neq 0 \\ \displaystyle\int_1^x \phi_\nu(t)\,dt & \text{if} \quad a_\nu = 0. \end{cases} \tag{2}$$

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