ON THE LOCAL BEHAVIOR OF HOMEOMORPHIC SOLUTIONS OF BELTRAMI'S EQUATION

By A. Schatz

1. Introduction. Let G be a bounded domain in the complex z plane and let $a(z)$ be a measurable function defined in G with $\sup |a(z)| = k < 1$. In this paper we shall study the local behavior of homeomorphic solutions, $w = w(z)$, of Beltrami's equation

$$w_{\bar{z}} = a(z) w_z \quad \text{in} \quad G. \tag{1.1}$$

By a solution we mean a function $w = w(z) \in L_1^{\text{loc}}(G)$, having weak formal derivatives w_z and $w_{\bar{z}}$ which are $L_p^{\text{loc}}(G)$ for some $p > 2$ and satisfy (1.1) almost everywhere in G. Since $w_{\bar{z}} \in L_p^{\text{loc}}(G)$ for some $p > 2$, it follows that $w \in C_\lambda$ on every compact subset of G, where $\lambda = (p - 2)/p$. Homeomorphic solutions are called quasiconformal mappings of G with complex dilatation $a(z)$. By definition every solution which we shall consider satisfies some Hölder condition inside G, and without further restrictions on the coefficient $a(z)$ it is difficult to make more precise statements about the pointwise behavior at or near a given point. In this direction many results have been obtained by various authors using additional restrictions on $a(z)$. For an extensive bibliography, the reader is referred to [4]. Those results which are of interest to us will be listed in the sections to follow, and here we shall confine ourselves to a description of the problems to be undertaken.

More precisely we shall seek to explicitly determine the essential behavior of homeomorphic solutions in the neighborhood of a given point under various assumptions on both the form and regularity of $a(z)$. The conditions we shall require are motivated by problems which arise in analysis and geometry.

In §§2 through 6, the conditions we impose on $a(z)$ do not imply that solutions are differentiable everywhere in a neighborhood of the point to be investigated, and therefore we first focus our attention on the zero-th order behavior. In §2 we prove a result which implies total differentiability at a point. Here our assumption (2.4) on $a(z)$ is very similar, but slightly more restrictive, than Lehto's, and correspondingly our result is slightly stronger. Our methods of proof differ, and the stronger result allows us to easily discuss the behavior of derivatives of solutions in angle in a different situation (see §7). In §3 we explicitly construct, in closed form, a homeomorphic solution of equation (1.1) in the case where $a(z)$ depends only on the angular variable θ. In §4 we show, using the explicit construction of the previous section, that as a result of the property that the complex dilatation $a(z)$ is in general not invariant under co-

Received February 16, 1967. This research was partially supported by a National Science Foundation contract number GP-5882.