ON COMPLETING THE VON NEUMANN ALMOST PERIODIC FUNCTIONS

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1. Introduction and history. Besicovitch [1] showed that the space of Bohr almost periodic functions could be completed by closing it with respect to the norm

$$||f||_{B^p} = \overline{\lim_{T\to\infty}} \left[\frac{1}{2T}\int_{-T}^{T} |f(x)|^p \, dx\right]^{1/p}, \qquad p \geq 1.$$

In the resulting function spaces $\{B^p - AP\}$ one identifies functions whose "distance" from one another is zero. The members of $\{B^p - AP\}$ have Fourier series and the spaces $\{B^p - AP\}$ closely resemble the more familiar L^p-spaces. In fact, Følner [6] has shown that the normed linear spaces $(\{B^p - AP\}, || \; ||_{B^p})$ and $(L^p(\bar{R}), || \; ||_p)$ are naturally isomorphic and isometric, where $\bar{R}$ is the Bohr compactification of the real line.

Let G be a locally compact T_0-topological group (= LC group) and $\alpha(G)$ the space of continuous complex-valued von Neumann almost periodic functions on G. It is natural to ask whether or not $\alpha(G)$ can be completed in a fashion which generalizes, or is at least analogous to, the Besicovitch procedure. Other authors have considered this question. In 1957, Følner [7] showed that if G is discrete, one may define a norm $|| \; ||_{B^p}^F$ on the set of all complex-valued functions on G such that the closure of $\alpha(G)$ with respect to $|| \; ||_{B^p}^F$ is complete, $p \geq 1$. The resulting space of functions is naturally isomorphic and isometric to the space $L^p(\bar{G})$, where $\bar{G}$ is the Bohr compactification of G. Unfortunately, however, the original Besicovitch spaces cannot be realized via the Følner procedure. Also the Følner norm $|| \; ||_{B^p}^F$ is defined through a quite complicated limiting process.

In 1958 Hirschfeld [10] considered LC groups G which have "left sampler families" $\{U_t\}_{t \varepsilon R}$ (cf., [13]). Roughly, these are families of open bounded (i.e., each $\bar{U}_t$ is compact) subsets of G satisfying enough conditions to insure that for all $f \, \varepsilon \, \alpha(G)$

$$Mf = \lim_{t\to\infty} \frac{1}{\mu(U_t)} \int_{U_t} f \, d\mu.$$

Here Mf denotes the mean value of f and μ is left Haar measure. One then closes $\alpha(G)$ with respect to the norm

$$||f||_{B^p}^H = \overline{\lim_{t\to\infty}} \left[\frac{1}{\mu(U_t)}\int_{U_t} |f|^p \, d\mu\right]^{1/p} \qquad p \geq 1.$$

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