A HOMOLOGY TRANSGRESSION THEOREM

By Allan Clark

In this note we prove a homology transgression theorem which is dual to the well-known cohomology transgression theorem of Kudo [4]. A similar theorem has been proved by Dyer and Lashof [2, Theorem 4.7] under very strong hypotheses of homotopy commutativity.

We shall assume throughout that X is a pathwise connected and simply connected H-space with homology of finite type. ΠX will denote the space of paths on X beginning at the base point e (which we choose to be the unit of the H-space product on X), and ΩX will denote the space of loops on X based at e. Then ΩX is the fiber of the path space fibering $\pi : \Pi X \rightarrow X$ which sends each path $\lambda : [0, 1] \rightarrow X$ into its terminal point $\lambda(1) = \pi(\lambda)$.

$\{E^r(\pi; Z_p)\}$ will denote the Serre spectral sequence in homology mod p (a prime) of the fibering π. Under the product induced by the H-space structure of X, π is a multiplicative fibering, and therefore $\{E^r(\pi; Z_p)\}$ is a spectral sequence of Hopf algebras over Z_p. We recall that $E^2_{p,q}(\pi; Z_p) \approx H_p(X; Z_p) \otimes H_q(\Omega X; Z_p)$ and remark that this isomorphism is as Hopf algebras. $\{u \otimes v\}$ in various elements of the spectral sequence will denote the appropriate iterated homology class of $u \otimes v \in E^2(\pi; Z_p)$ where $u \in H_*(X;(X; Z_p)$ and $v \in H_*(\Omega X; Z_p)$.

Theorem 1. *If $x \in H_{2m}(X; Z_p)$ transgresses to $y \in H_{2m-1}(\Omega X; Z_p)$, then*

1) $x^p \in H_{2mp}(X; Z_p)$ *transgresses to* $y' \in H_{2mp-1}(\Omega X; Z_p)$;
2) $\beta x^p \in H_{2mp-1}(X; Z_p)$ *transgresses to* $w \in H_{2mp-2}(\Omega X; Z_p)$;
3) $d^{2m(p-1)}\{x^{p-1} \otimes y\} = \{1 \otimes (\beta y' + w)\} \in E^{2m(p-1)}_{0,2mp-2}(\pi; Z_p)$.

(*Here β denotes the* mod p *homology Bockstein operator.*)

1. The cobar construction. Since the main part of our proof of Theorem 1 involves computations in the cobar construction of a mod p chain complex for X, we recall the cobar construction as described by Adams.

Let K be a commutative ring with unit and let $\otimes$ denote $\otimes_K$. Let C denote a *connected differential graded K-coalgebra.* This means:

1) C is a differential graded K-module in which $C_0 \approx K$ and the differential $d_1 : C_1 \rightarrow C_0$ is zero. Then as a differential graded K-module $C \approx \bar{C} \oplus K$ where $\bar{C}_n = C_n$ for $n > 0$ and $\bar{C}_0 = 0$.

2) C is a K-coalgebra with associative coproduct $\Delta : C \rightarrow C \otimes C$ which is a morphism of differential graded K-coalgebras.

3) The augmentation $\epsilon : C \rightarrow K$ (which coincides with the projection $\bar{C} \oplus$

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