A SUMMABILITY THEOREM IN FOURIER SERIES

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1. Introduction. It was pointed out by Salem and Zygmund in [2] that there exist periodic functions $f(x)$ in the Lipschitz class λ_α, $0 < \alpha < 1$ such that if the Fourier series $S[f] = \sum_0^\infty A_n(x)$, then $\sum_0^\infty n^\alpha A_n(x)$ is not summable by Abel's method in any set E of positive measure. In this paper we prove the following theorem:

Theorem A. *Let $f(x)$ be a periodic, integrable function of period 2π. Suppose, for $0 < \alpha < 1$*

(1) $f(x + t) = f(x) + o(|t|^\alpha)$ *as* $t \to 0$

$$\lim_{\epsilon \to 0} \int_\epsilon^\infty \frac{f(x + t) + f(x - t) - 2f(x)}{t^{1+\alpha}}\, dt \quad \textit{exists} \tag{2}$$

for x in a set E of positive measure. Then $\sum_0^\infty n^\alpha(a_n \cos nx + b_n \sin nx)$ is summable (C, α) a.e. in E.

Besides being of interest in Fourier series, this theorem serves to illustrate two additional points. In the first place it is proved by using the methods of repeated convergence classes (see [1]) and thus provides a concrete example of where these methods can successfully be applied. Secondly, in [1] classes $R_{\delta\beta}$ and $F_{\delta\beta}$ are considered (see Definition 1 in §2) where $R_{\delta\beta} = F_{\delta\beta}$ when δ is an integer and $R_{\delta\beta}$ is distinct from $F_{\delta\beta}$ when δ is fractional. It was also indicated in [1] that the properties enjoyed by the class $R_{\delta\beta}$, δ an integer, are shared between the two classes for δ nonintegral. Thus, in many instances, a theorem which is true for $\sum a_n \in R_{\delta\beta}$, δ integral, and false when δ is fractional, becomes valid again if in the fractional case we assume, in addition, that $\sum a_n \in F_{\delta\beta}$. The above theorem illustrates this idea. If $\alpha = 1$, then $f(x + t) = f(x) + f'(t) + o(t)$ already implies that

$$\lim_{\epsilon \to 0} \int_\epsilon^\infty \frac{f(x + t) + f(x - t) - 2f(x)}{t^2}\, dt$$

exists, and the theorem reduces to a well-known classical result. Zygmund and Salem showed that $f(x + t) = f(x) + o(|t|^\alpha)$, $0 < \alpha < 1$ no longer implies that $\sum n^\alpha A_n(x)$ is summable (C, α). However, if we assume that

$$\lim_{\epsilon \to 0} \int_\epsilon^\infty \frac{f(x + t) + f(x - t) - 2f(x)}{t^{1+\alpha}}\, dt$$

exists, which, by Lemma 2 below is closely related to the Fourier series of $f(x)$

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