SETS OF VALUES OF GENERAL DIRICHLET SERIES

By Robert Spira

1. Introduction. Harald Bohr [1] gave four theorems, stated below, on the set of values of a general Dirichlet series,

$$(1) \qquad f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \qquad \lambda_{n+1} > \lambda_n , \quad \lambda_n \to +\infty ,$$

taken along a given vertical line, and the relation of this set to the set of values assumed in the neighborhood of the line and also to the set of values of a companion function of countably many variables. In this paper, we generalize Bohr's theorems to sets of lines, restricted in some cases to have compact or open projections on the real axis. These strengthened forms of Bohr's theorems are necessary in the applications [2].

2. Bohr's theorems. Let L be a finite or countable set of real numbers $\lambda_1 , \lambda_2 , \cdots$ and let R_L be the vector space over the rationals generated by L. We say B is a *basis of* L, provided it is a basis of R_L . For such a B, each element of L is a linear combination of elements of B with rational coefficients, and if these rational coefficients are integral, we say B is an *integral basis of* L.

Let $f(s)$ be as (1) above and let $B = \{\beta_1 , \beta_2 , \cdots \}$ be a basis for $\{\lambda_1 , \lambda_2 , \cdots \}$. Thus, we have

$$(2) \qquad \lambda_n = r_{n,1}\beta_1 + r_{n,2}\beta_2 + \cdots + r_{n,q_n}\beta_{q_n} ,$$

with rational $r_{n,i}$. We define,

$$(3) \qquad F(x_1 , x_2 , \cdots) = \sum_{n=1}^{\infty} a_n \exp\left(-(r_{n,1}x_1 + \cdots + r_{n,q_n}x_{q_n})\right).$$

Let A be the abscissa of absolute convergence of $f(s)$ and let $\sigma_0 > A$. As in Bohr's paper, we define,

$$(4) \qquad V(\sigma_0) = \text{the set of function values } f(\sigma_0 + it), \; -\infty < t < \infty,$$

$$(5) \qquad W(\sigma_0) = \bigcap_{\delta>0} \left\{ \begin{matrix} \text{set of function values of } f(\sigma + it), \\ \sigma_0 - \delta < \sigma < \sigma_0 + \delta, \; -\infty < t < \infty \end{matrix} \right\},$$

$$(6) \qquad \begin{matrix} U(\sigma_0) = \text{the set of values of } F(x_1 , x_2 , \cdots) \text{ as each} \\ x_j \text{ independently runs over the line } \mathrm{Re}\, x_j = \beta_j\sigma_0 . \end{matrix}$$

Finally, we define two Dirichlet series with the same λ_i's, $\sum a_n e^{-\lambda_n s}$ and

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