NOTE ON G. WHAPLES' PAPER "ALGEBRAIC EXTENSIONS OF ARBITRARY FIELDS"

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This note engages in a question dealt with in [1, Chapter IX], [2; 93–95] and [3]. The denotations are as in [3], and all fields have characteristic $\neq 2$. In the proof of Case 3, Theorem 2, Whaples uses the wrong assumption that $k(\zeta_{n+\nu})$ has an automorphism τ for which $\zeta_{n+\nu}^{\tau} = \zeta_{n+\nu}^{-1}$; this argument is repeated towards the end of Case 5, but is correct there as the corollary to Lemma 1 shows. This is not so in Case 3, for the existence of the fixed fields C_ν depends on the existence of the subgroup $\{\tau, \tau^2 = id\}$ of the Galois-group $G(k(\zeta_{n+\nu})/k)$ which, in general, does not exist; for one easily verifies the following lemma.

Lemma 1. *If the proper extension $k(\zeta_2)$ of k contains ζ_m and $N_{k(\zeta_2)/k}(\zeta_m) = -1$, then the automorphism τ mapping ζ_2 to $-\zeta_2$ cannot be extended to $k(\zeta_{m+1})$ such that $\tau^2 = id$.*

Corollary. *If $\zeta_2 \notin k$ and $k(\zeta_2)$ contains all ζ_ν, then $N_{k(\zeta_2)/k}(\zeta_\nu) = 1$ for all ν.*

This situation actually occurs as the simple example $k = Q((-2)^{1/2})$ shows: (i) k does not contain $\zeta_2 = i$, (ii) $k(i)$ contains ζ_2 and $\zeta_3 = (-2)^{-1/2} - (-2)^{-1/2}i$, but not ζ_4, (iii) $N_{k(i)/k}(\zeta_3) = -1$. Fortunately, Lemma 1 implies the following lemma which permits a correct proof of Case 3.

Lemma 2. *If k is a field without ζ_2 and $k(\zeta_2)$ contains ζ_n but not ζ_{n+1}, then $G(k(\zeta_{n+\nu})/k)$ is cyclic of degree $2^{\nu+1}$ if and only if $N_{k(\zeta_2)/k}(\zeta_n) = -1$.*

We need only to prove one direction and assume that $N_{k(\zeta_2)/k}(\zeta_n) = -1$. If $G = G(k(\zeta_{n+\nu})/k)$, $\nu \geq 1$, were not cyclic, then $G = G(k(\zeta_{n+\nu})/k(\zeta_2)) \oplus H$ with $H \cong Z/2Z$. Hence, G would contain an element τ of order 2 which is not in $G(k(\zeta_{n+\nu})/k(\zeta_2))$, contradicting Lemma 1. Thus, we split the proof of Case 3 for $p = 2$ into two parts: If $N_{k(\zeta_2)/k}(\zeta_n) = 1$, then Whaples' proof applies; if $N_{k(\zeta_2)/k}(\zeta_n) = -1$, then the above lemma settles the matter.

References

1. A. A. Albert, *Modern Higher Algebra*, Chicago, 1937.
2. E. Artin and J. Tate, *Class Field Theory*, Harvard, 1961.
3. G. Whaples, *Algebraic extensions of arbitrary fields*, this Journal, vol. 24(1957), pp. 201–204.

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