# A LOWER BOUND IN ASYMPTOTIC DIOPHANTINE APPROXIMATIONS

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**1. Introduction.** Let $\alpha$ be a real irrational number. Define for all real numbers $B \geq 1$, $\lambda(B, \alpha)$ to be the number of solutions in integers $p$, $q$ of the inequalities

$$(1) \qquad |q\alpha - p| < \frac{1}{q} \quad \text{and} \quad 1 \leq q \leq B.$$

Then it is known [7] that for almost all $\alpha$,

$$\lambda(B, \alpha) \sim 2 \log B \qquad (B \to \infty).$$

It is not hard to show that there is a universal constant $c$ such that for all irrationals $\alpha$

$$\lambda(B, \alpha) \geq c \log B + O(1) \qquad (B \to \infty).$$

The purpose of this paper is to find the best possible value of $c$. This turns out to be

$$c_0 = 3 \Big/ \log \left( \frac{9 + \sqrt{77}}{2} \right) = 1.373 \cdots .$$

We show that if $\alpha$ is equivalent to

$$\alpha_0 = [0, 1, 7, 1, 7, 1, 7, \cdots] = \frac{-7 + \sqrt{77}}{2},$$

then

$$\lambda(B, \alpha) = c_0 \log B + O(1).$$

($\alpha_0 = [0, 1, 7, 1, 7, 1, 7, \cdots]$ denotes the continued fraction expansion of $\alpha_0$. We say two irrationals are equivalent if the tail ends of their continued fractions are the same.) But also there are uncountably many irrationals $\alpha$ such that

$$\lambda(B, \alpha) \sim c_0 \log B.$$

In order to obtain these results we use the theory of continued fractions (see [3, 5]). The first job (in §2) is to show how to count in terms of continued fractions. This was done in [1] but here we need a more explicit answer. We also need to know $\lambda(B, \alpha)$ if $\alpha$ is a quadratic irrationality. Lang [4], [5] showed that if $\alpha$ is quadratic, then there is a constant $c = c(\alpha)$ such that

$$\lambda(B, \alpha) = c \log B + O(1).$$

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