SELF-RECIPROCAL FUNCTIONS FOR THE HANKEL TRANSFORMATION OF INTEGER ORDER

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If $f(x)$ belongs to $L^2(0, \infty)$, its Hankel transform of order $\nu > -1$ is defined by

$$g(x) = \int_0^\infty f(t) J_\nu(xt)(xt)^{\frac{1}{2}}\, dt,$$

where

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}x)^{2n+\nu}}{n!\, \Gamma(n + \nu + 1)}$$

is the Bessel function of order ν. The integral converges in the mean square sense, and the linear transformation $f(x) \to g(x)$ so defined in $L^2(0, \infty)$ is both isometric and selfadjoint, and hence equal to its own inverse. A self-reciprocal function is a function which is equal to its own Hankel transform. A function is called skew-reciprocal if it is equal to minus its own Hankel transform. By an example of Sonine [6], it is possible for a self-reciprocal or skew-reciprocal function to vanish in an interval $(0, a)$ without vanishing identically. In the case $\nu = 0$, a study of this phenomenon has led to an interesting isometric expansion which exhibits all Hankel transform pairs in $L^2(0, \infty)$ vanishing in $(0, a)$. This expansion was first obtained by de Branges [2], and later by V. Rovnyak [5]. In this paper we extend the methods of [5] to give a similar partially isometric expansion when ν is a positive integer.

Throughout the paper we assume that ν is a positive integer, but Theorems I and II and the lemma to Theorem VI are true as stated for arbitrary $\nu > 0$.

As in [5] we use a device due to Hardy and Titchmarsh [4]. We begin by stating this in a convenient form. Let $\mathfrak{D}^\nu$ be the Hilbert space of analytic functions $F(z)$ defined for $y > 0$, such that

$$\|F\|^2 = \Gamma(\nu)^{-1} \int_0^\infty \int_{-\infty}^\infty |F(x + iy)|^2\, y^{\nu-1}\, dx\, dy < \infty.$$

Theorem I. *If $f(x)$ belongs to $L^2(0, \infty)$, then*

$$F(z) = \int_0^\infty t^{\nu+\frac{1}{2}} e^{\frac{1}{2}izt^2} f(t)\, dt, \qquad y > 0, \tag{1}$$

belongs to $\mathfrak{D}^\nu$ and

$$\|F\|^2 = 2\pi \int_0^\infty |f(t)|^2\, dt. \tag{2}$$

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