# HANKEL TRANSFORMS OF ARBITRARY ORDER

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**1. Introduction.** It is well known that the Hankel transformation of order $\mu \geq -\frac{1}{2}$:

$$F(y) = (\mathfrak{H}_\mu f)(y) = \int_0^\infty f(x)\sqrt{xy}\, J_\mu(xy)\, dx, \tag{1}$$

where $J_\mu$ is the Bessel function of first kind and order $\mu$ and $f(x)$ is a suitably restricted function, is inverted by the same formula:

$$f(x) = (\mathfrak{H}_\mu^{-1} F)(x) = \int_0^\infty F(y)\sqrt{xy}\, J_\mu(xy)\, dy. \tag{2}$$

In symbols, $\mathfrak{H}_\mu^{-1} = \mathfrak{H}_\mu$ if $\mu \geq -\frac{1}{2}$ [3; §14.4]. J. L. Lions [2] has extended these formulas to all real and complex values of $\mu$ excluding $\mu = -1, -2, -3, \cdots$ in such a way that they have a meaning for certain classes of Schwartz distributions.

In this paper we present another method of extending the Hankel transformation to negative real values of the order $\mu$. The resulting extension has the following properties: (i) The generalized direct transformation $\mathfrak{H}_\mu'$ possesses an inverse $(\mathfrak{H}_\mu')^{-1}$ for every real value of $\mu$. (ii) $\mathfrak{H}_\mu'$ and $(\mathfrak{H}_\mu')^{-1}$ are defined on certain classes of generalized functions, which were developed in [4]. (iii) If $\mu \geq -\frac{1}{2}$ and the generalized function, which is being transformed, is a suitably restricted function, then the direct and inverse transformations coincide with (1) and (2) respectively.

**2. Terminology and notation.** Throughout this work, $I$ denotes the one-dimensional real open interval $(0, \infty)$, and $x$ and $y$ are variables in I. As usual, $L_1(I)$ is the space of (equivalence classes of) Lebesgue-integrable functions on $I$. By a smooth function we mean a function that possesses continuous derivatives of all orders at every point of its domain. If $f$ is a generalized function on $I$, the notation $f(x)$ is used merely to indicate that the testing functions on which $f$ is defined possess $x$ as their independent variable. The symbol $\langle f, \phi \rangle$ denotes the number assigned to some element $\phi$ in a testing function space by a member $f$ of the dual space. The $k$-th derivative of an ordinary or generalized function $f(x)$ is denoted alternatively by $D^k f$, $D_x^k f(x)$, or $f^{(k)}(x)$. When we refer to a mapping on a topological linear space as an isomorphism or automorphism, it is understood that the continuity of the mapping and its inverse is also implied.

We shall adopt the following notational convention throughout this paper. When a lower case (respectively, capital) letter is used to denote a testing

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