ARCHIMEDEAN QUOTIENT RIESZ SPACES

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1. Introduction. For the definitions of the basic properties of *Riesz spaces*, also called vector lattices, we refer to [4].

Let L be a Riesz space and let $A \subset L$ be a linear subspace of L. Then we recall that A is said to be an *ideal* whenever $|g| \leq |f|$ and $f \in A$ implies $g \in A$. If A is an ideal of a Riesz space L and π denotes the quotient map of L onto L/A, then L/A is again a Riesz space under the ordering: $\pi(f) \geq 0$ whenever $f + u \geq 0$ for some $0 \leq u \in A$. A Riesz space L is said to be *Archimedean* whenever $nu \leq v$ $(n = 1, 2, \cdots)$ and $0 \leq u, v \in L$ implies $u = 0$.

In this paper we shall investigate some aspects of the theory of quotient spaces of Riesz spaces. In particular, we shall give an answer to the question of when a quotient space of a Riesz space is Archimedean. The solution of this problem can be given in the following form: L/A is Archimedean if and only if A is closed in the relative uniform topology.

In §2, 3, and 4 we gather together several facts on the notion of relative uniform convergence and the relative uniform topology which it determines. The notion of sequential relative uniform completeness is defined, and it is shown that the quotient space of a s.r.u.-complete Riesz space is s.r.u.-complete. In §5 we then prove the main theorem quoted above—that a Riesz homomorphic image of a Riesz space is Archimedian if and only if its kernel is closed in the relative uniform topology. §6 is concerned with the properties of arbitrary Riesz spaces in terms of their Archimedean quotient spaces. In the final section we give a complete characterization of those Riesz spaces which have the property that every quotient space is Archimedean. It turns out that they are essentially the Riesz spaces of all real functions which vanish off finite subsets of a given set.

2. Relative uniform convergence. In 1912 E. H. Moore [5] introduced the notion of *relative uniform convergence* for sequences of real functions defined on a given set. It is, however, also a notion of convergence which can be defined naturally within the framework of the theory of Riesz spaces (see also [6], "uniform convergence").

Definition 2.1. *A sequence* $\{f_n : n = 1, 2, \cdots\}$ *of elements of a Riesz space* L *is said to converge relatively uniformly to an element* f, *written* $f_n \rightarrow f$(r.u.),

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