ON ORBITS IN SPACES OF INFINITE MEASURE

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1. Introduction. Our purpose here is to extend some results of Maharam [6] from the unit interval [0, 1] to the real line. Since our results are analogous to those of [6], proofs are omitted except where the argument is substantially different from that for the corresponding result in [6]. Accordingly, the reader will be assumed to be familiar with [6].

Briefly, the problem is to discover conditions on a sequence of points which guarantee that it shall "determine" an essentially unique ergodic measure-preserving transformation T, and which are, conversely, possessed by the positive semi-orbit of almost every point under such a transformation. This was solved for [0, 1] by Maharam, who made essential use of both the compactness and finiteness of measure. The relaxation of the requirements is reflected here in two ways. First, a $1 - 1$ continuous function on a compact set has a continuous inverse—this is not true in the non-compact case, forcing us to consider backward, as well as forward, shifts (see §2 for pertinent definitions). Second, a measure-preserving transformation T on a σ-finite measure space is ergodic if and only if it induces an ergodic transformation T_E on some measurable set E with $0 < \mu(E) < \infty$. To deal with induced transformations, we need to consider T^k for positive integers k—this in turn requires consideration of k-shifts, whereas in [6], only the forward 1-shift was needed.

In §2, we establish notation and state the required preliminary results. §3 contains the definitions of relatively uniformly distributed D^*- and I^*-sequences, and the theorems showing that these determine essentially unique measure-preserving and incompressible transformations, respectively. In §4, we define E^*-sequences and show that each such sequence determines an ergodic measure-preserving transformation, and that conversely, the positive semi-orbit of almost every point under such a transformation is an E^*-sequence. §4 also contains an adaptation of an example of Maharam which shows that something more than relative uniform distribution is needed to guarantee ergodicity; indeed, we construct a relatively uniformly distributed D^*-sequence which determines the identity a.e..

The results proved here extend to euclidean n-space with no more than notational changes, but the question of extension to more general locally compact spaces of σ-finite measure is open. The topological requirements for these proofs are stingent enough to guarantee that the space be metric, and we also need to know that the closure of each open sphere is a) compact, and b) the corresponding closed sphere. Some conditions which imply a) are known (see,

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