INCIDENCE FUNCTIONS AS GENERALIZED ARITHMETIC FUNCTIONS, I.

By David A. Smith

1. Introduction. An old idea, now known as the incidence algebra of a locally finite partially ordered set, was recently resurrected by G.-C. Rota [10] and proposed as the basis for a new study of combinatorial theory from a unified point of view. Much of Rota's paper is devoted to demonstrating the importance of the "Möbius function" of the partially ordered set, the inverse in the incidence algebra of the characteristic function of the order relation.

The prime example of an incidence algebra is that for which the partially ordered set is the set $\mathbf{N}$ of natural numbers, ordered by divisibility. The algebra itself consists of functions from $\mathbf{N} \times \mathbf{N}$ into a field such that $f(x, y) = 0$ unless $x \mid y$. The operations are pointwise addition and scalar multiplication and a convolution (Dirichlet) type product. The subalgebra of functions whose values depend only on the quotient of the arguments is isomorphic to the ordinary algebra of arithmetic functions, with the Dirichlet product. The Möbius function is this case is just the classical Möbius function. The algebra of arithmetic functions has many nice properties. This paper is devoted to the study of those properties of arithmetic functions which can be extended to certain incidence algebras.

In the next section we list some of the elementary properties of incidence functions, particularly concerning inversion. In §3, we give an axiomatic characterization of incidence algebras, analogous to that given by Carlitz [3] for the algebra of arithmetic functions.

§§4 and 5 are concerned with incidence algebras on lattices, for which we introduce notions of factorability and additivity of functions which generalize the corresponding notions for arithmetic functions. We show that the factorable functions form a subgroup of the multiplicative group of invertible elements if and only if the underlying lattice is distributive. Also, the factorability or additivity of certain special functions is seen to be closely related to the structure of the lattice.

In §6, we introduce some more special functions, including a generalized Euler ϕ-function, and list some of their properties which are analogous to properties of arithmetic functions.

§7 contains an application of some of the results to arithmetic functions themselves. Specifically, it is shown that not only the Dirichlet product, but also the Cauchy, unitary [7], [8], [11], Lucas [4] products of arithmetic functions can be viewed as incidence algebra products for suitable partial orderings of $\mathbf{N}$

Received August 23, 1966. Supported in part by National Science Foundation grant GP 4473.