MULTIRESTRICTED AND ROWED PARTITIONS

By M. S. Cheema and C. T. Haskell

Interconnections are established between various multirestricted partition numbers; proofs are given using analytical and combinatorial methods. Certain results are established for rowed partition numbers $t_r(n)$ defined by

$$\prod_{n=1}^{\infty} (1 - x^n)^{-\min(n,r)} = \sum_{n=0}^{\infty} t_r(n)x^n,$$

and asymptotic formulas are obtained for $p_{-r}(n)$ defined by

$$\prod_{n=1}^{\infty} (1 - x^n)^{-r} = \sum_{n=0}^{\infty} p_{-r}(n)x^n.$$

The series for $p_{-r}(n)$ are generalizations of the Hardy–Ramanujan and Rademacher asymptotic series for $p(n)$ the number of partitions of n.

1. Introduction and statement of main results. Let $q(n, r)$ denote the number of partitions of the non-negative integer n into at most r parts (by definition, $q(0, r) = 1$; other functions to be defined will also yield 1 at $n = 0$). This function has been investigated extensively, and tables exist for evaluating it. (See, for example, Gupta et al. [3], in which $p(n, r)$ is used where we use $q(n, r)$.) Its generating function is given by

$$\prod_{j=1}^{r} (1 - x^j)^{-1} = \sum_{n=0}^{\infty} q(n, r)x^n. \tag{1.1}$$

Let $v(n, r; m)$ denote the number of partitions of n into at most r parts, each part $\leq m$. MacMahon [10] has shown the generating function to be

$$\prod_{j=1}^{r} \frac{1 - x^{m+j}}{1 - x^j} = \sum_{n=0}^{\infty} v(n, r; m)x^n. \tag{1.2}$$

(Actually, the summation in (1.2) must be finite, since $v(n, r; m) = 0$ for all $n > rm$).

We now introduce some functions for multirestricted partitions, each of which can be evaluated in terms of one of the two prior-known functions mentioned above. The number of partitions of n into r parts, each part $\geq m$, will be denoted by $p(n, r; m)$. $p'(n, r; m)$ (and other functions $P'(n, r; m)$ etc.) will designate the additional restriction of distinct parts; if the parts are required to be odd, the function will be capitalized (in this case, m will be understood to be odd); if the parts must not exceed k, k will be introduced as a parameter.

Received August 2, 1966; in revised form, November 2, 1966. These results form a part of the dissertation of the second-named author at the University of Arizona and were supported in part by NSF grant GP-4217.