ON CONVOLUTION AND FOURIER SERIES

By Jack Bryant

1. Introduction. Consider the convolution $g * h$ of integrable periodic functions g and h in $L(0, 2\pi)$ defined by

$$g * h(x) = \frac{1}{\pi} \int_0^{2\pi} g(x - t)h(t)\, dt.$$

If f is a given function in $L(0, 2\pi)$ satisfying "suitable" conditions, when is it possible to write $f = g * h$, with $g \in L(0, 2\pi)$ and h satisfying the same "suitable" conditions? We answer this question for some interpretations of "suitable", extending a theorem of Salem [1; 108–14] in which "suitable" means "$f \in L_p$, $1 \leq p < \infty$" or "$f \in C[0, 2\pi]$".

We base all our results on a rather tedious calculation; this is presented in §2, along with some estimates and a lemma on $(C, 1)$ summability of series. In §3, we furnish at least a partial answer to the above question, and point out connections with $(C, 1)$ summability of Fourier series.

We conclude our introductory remarks by introducing the following notation: If f is a function, $\tau_h f$ is the function defined by $(\tau_h f)(x) = f(x + h)$. B will denote a Banach space with norm $||\cdot||$. If $f \in L$, then $S[f] = \sum A_n$ denotes the Fourier series of f, $\{s_n(x)\}$ the partial sums and $\{\sigma_n(x)\}$ the $(C, 1)$ means of $S[f]$. If $\{\lambda_n\}$ is a sequence, we write $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$ and $\Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$. "$A$" will denote an absolute constant, not always the same each time it appears.

2. The principal calculations. Let $\{A_n\}$ be a sequence of elements of a linear space, f any element, and $\{\lambda_n\}$ be a sequence of numbers. Suppose $s_n = \sum_0^n A_k$, $t_n = \sum_0^n \lambda_k A_k$, and let σ_n and τ_n be the $(C, 1)$ means of $\sum A_k$ and $\sum \lambda_k A_k$, respectively. Then, summing by parts twice, we obtain

$$\begin{aligned}
(n + 1)\tau_n &= \textstyle\sum_0^n t_k \\
&= \textstyle\sum_0^{n-1} (n + 1 - k)\, \Delta\lambda_k s_k + \sum_0^{n-1} \lambda_{k+1} s_k + \lambda_n s_n \\
&= \textstyle\sum_0^{n-2} (n + 1 - k)\, \Delta^2\lambda_k (k + 1)\sigma_k + \sum_0^{n-2} \Delta\lambda_k (k + 1)\sigma_k \\
&\quad \textstyle + 2\, \Delta\lambda_{n-1} n\sigma_{n-1} + \sum_0^{n-2} \Delta\lambda_{k+1}(k + 1)\sigma_k + (n + 1)\lambda_n\sigma_n \\
&= \textstyle\sum_0^{n-1} (n + 1 - k)\, \Delta^2\lambda_k (k + 1)\sigma_k + 2 \sum_0^{n-1} \Delta\lambda_{k+1}(k + 1)\sigma_k \\
&\quad + (n + 1)\lambda_n\sigma_n .
\end{aligned}$$

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