APPROXIMATION OF HARMONIC FUNCTIONS OF THREE VARIABLES BY HARMONIC POLYNOMIALS

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1. Introduction. The question of approximation of harmonic functions of three variables by harmonic polynomials in simply connected domains is of considerable interest, in particular, when solving a boundary value problem and using harmonic polynomials orthonormal in a simply connected domain b^3.

Results of this kind can be obtained in various ways, e.g., by those indicated in I and II.

I. One uses the integral operator

$$\mathbf{B}_3(f(u, \zeta)) \equiv \frac{1}{2\pi i} \int_{|\zeta|=1} f(u, \zeta) \frac{d\zeta}{\zeta} = H(x, y, z) \tag{1}$$

transforming analytic functions of *two* complex variables $u = x + \zeta(iy + z)/2 + \zeta^{-1}(iy - z)/2$ and ζ into harmonic functions of three variables. Using the theorems about approximation of $f(u, \zeta)$ by polynomials and the representation of f in terms of H, one obtains theorems about the degree of approximation. See [7; 43].

The theory of approximation of functions of one complex variable is developed to a larger extent than that of two complex variables. Consequently it is useful to determine subclasses of harmonic functions $H(x, y, z)$ whose $\mathbf{B}_3$-associates $f(u, \zeta)$ are functions of one complex variable u multiplied by a *fixed* factor $l(\zeta)$. Using then results by Walsh [19], Mergeljan [16] and others for analytic functions of *one* complex variable, one obtains theorems on the degree of approximation of functions H belonging to these subclasses of harmonic functions.

Remark. The study of subclasses of harmonic functions possessing certain properties is of importance also for various other purposes. See [3], [4], [8], [10], [11], [12], [13], [17], [18].

In the present paper we set: $l(\zeta) = \zeta^{\kappa}$($\kappa$ an integer) and denote by $\mathbf{S}$ the class of functions generated by (1) with $f(u, \zeta) = p(u)\zeta^{\kappa}$. Here $p(u)$ is an arbitrary analytic function of u, regular at the origin, and κ is a fixed integer. In this special case we obtain for the inverse of $\mathbf{B}_3$ an expression different from that derived in [2], see also [7; 43]. Using this method of attack, in §2 of the present paper the problem of approximation (by polynomials) of functions $H(x, y, z) \in \mathbf{S}$ is reduced to the following question in the theory of functions $p(u)$ of one complex variable:

Suppose $p(u)$ is regular in a (simply connected) domain B^2, we have to deter-

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