INTEGRABILITY THEOREMS FOR FOURIER SERIES

By Richard Askey* and Stephen Wainger**

Hardy and Littlewood have proven the following theorem: See [3] or [6, Vol. II; 129].

Theorem A. *If* $a(1) \geq a(2) \geq \cdots$, $a(n) \to 0$, *a necessary and sufficient condition that the function*

$$g(x) = \sum_{n=1}^{\infty} a(n) \cos nx$$

be in L^p, $p > 1$, *is that the sum*

$$\sum (a(n))^p n^{p-2}$$

is finite.

We wished for reasons explained elsewhere to weaken the hypothesis that $a(n)$ be monotonic to a condition that $n^{-k}a(n)$ should be monotonic for some non-negative integer k; we also needed to consider certain weighted L^p norms. See [1]. We establish this result in Corollary 1 below. It struck us while working on this problem that the condition that $a(n)$ be monotonic (or that $a(n)n^{-k}$ be monotonic) seems odd in this day of norm inequalities. In Theorems 1 and 2 below we show how to change Theorem A to more general results with norm inequalities. We also prove our theorems with weighted L^p norms in order to obtain Corollary 1, and our theorems hold when $p = 1$. We thus generalize results of Boas [2] and Shah [5].

We have referred above to weighted L^p spaces. We say that $f(\theta)$ is in $L(p, \alpha)$ if

$$\int_0^{\pi} |f(\theta)|^p (\sin \theta)^{\alpha p} \, d\theta < \infty .$$

(We shall always assume $p < \infty$.) If $f(\theta)$ is in $L(p, \alpha)$, we define

$$||f||_{p,\alpha} = \left\{ \int_0^{\pi} |f(\theta)|^p (\sin \theta)^{\alpha p} \, d\theta \right\}^{1/p}.$$

Finally we remark that much of our reasoning depends on Hardy's inequality. Hardy's inequality asserts the following:

Let $f(\theta)$ be in $L(p, \alpha)$ with $p \geq 1$ and $\alpha p < p - 1$. Let

$$F(\theta) = \frac{1}{\theta} \int_0^{\theta} f(\phi) \, d\phi.$$

Received November 18, 1965. *Supported in part by N. S. F. Grant GP-3483.
**Supported in part by N. S. F. grant GP-1645.