RECURSIVELY GENERATED STURM-LIOUVILLE POLYNOMIAL SYSTEMS

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Introduction. A three-term recurrence relation

$$
\begin{aligned}
&\phi_0(x) = 1 \\
(1)\qquad &\phi_1(x) = A_0x + B_0 \\
&\phi_{n+1}(x) = (A_nx + B_n)\phi_n(x) - C_n\phi_{n-1}(x), \qquad n = 1, 2, 3, \cdots,
\end{aligned}
$$

where $A_0 \neq 0$ and $A_nC_n \neq 0$ ($n = 1, 2, 3, \cdots$), generates a sequence $\{\phi_n(x)\}$ of polynomials in which ϕ_n is of degree exactly n. Some (but not all) sequences so generated are Sturm-Liouville polynomial systems—that is, sequences $\{\phi_n(x)\}$ of polynomials in which, for each n, the n-th-degree polynomial ϕ_n is a solution of a differential equation of the form

$$(2)\qquad a_0(x)y'' + a_1(x)y' + [a_2(x) + \lambda_n]y = 0,$$

where λ_n is a parameter depending on n but not on x [1]. Since much is known about Sturm-Liouville systems (the Legendre polynomials $P_n(x)$, for example), and since (1) is of some interest [2], [3], [4], [5], [6], [8], [9], [10], it would be useful to have

(i) a straightforward procedure for constructing the differential equation (2) from the coefficients in (1) under the *supposition* that the polynomials generated by (1) are solutions of (2);

(ii) a simple criterion for deciding, once (2) is constructed, whether the polynomials generated by (1) *actually are* solutions of (2).

In the pages which follow, such a procedure and such a criterion are deduced, and in each only the coefficients A_n, B_n, C_n in (1) are used. In addition, it is shown that if $\lambda_i \neq \lambda_j (i \neq j;\ i, j = 0, 1, 2, \cdots)$, then polynomials for which the criterion holds belong to exactly one of four possible classes, the particular class being easily identified by methods which are described. Finally, two examples are given which illustrate the procedure and the criterion and which also indicate how additional information about the polynomials can be obtained from the differential equation.

The relevant differential equation. In trying to decide whether the polynomials ϕ_n generated by (1) are solutions of an equation such as (2), it suffices to consider only equations of the form

$$(3)\qquad (\gamma x^2 + \beta x + \alpha)y'' + (\epsilon x + \delta)y' + \lambda_ny = 0, \qquad ' \sim d/dx,$$

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