A FUNCTIONAL-INTEGRAL EQUATION WITH APPLICATIONS TO HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

Dedicated to the memory of Robert E. Fullerton, late Professor of Mathematics, University of Maryland

By A. K. Aziz

1. Introduction. Consider the system of integral equations:

$$\begin{aligned}\varphi(x) &= g(x) + \alpha(x)\psi(\Gamma_1(x)) + \int_{x^*}^{x} K_1(\xi, x)\varphi(\xi)\, d\xi + \int_{y^*}^{\Gamma_1(x)} K_2(\eta, x)\psi(\eta)\, d\eta \\ \psi(y) &= h(y) + \beta(y)\varphi(\Gamma_2(y)) + \int_{x^*}^{\Gamma_2(y)} K_3(\xi, y)\varphi(\xi)\, d\xi + \int_{y^*}^{y} K_4(\eta, y)\psi(\eta)\, d\eta,\end{aligned} \tag{1.1}$$

where g, h, α, β, Γ_1, Γ_2, K_i, $i = 1, 2, 3, 4$ are given real-valued functions. Let $R = I_1 \times I_2$, $I_1 = \langle 0, r_1 \rangle$, $I_2 = \langle 0, r_2 \rangle$, where r_1, $r_2 > 0$ and x^*, $y^* \in R$. Let C_1 and C_2 be two curves with equation $y = \Gamma_1(x)$ and $x = \Gamma_2(y)$ respectively and lying entirely in R.

In the present paper we propose to find a pair of continuous functions $(\varphi(x), \psi(y))$ which satisfy (1.1). (For a precise statement of the smoothness assumptions of the functions involved see the statements of the theorems in §2.) Systems of the type (1.1) often occur in connection with boundary value problems for linear and non-linear differential equations of hyperbolic type, see for example [1], [2], [3], [5]. In §2 we give three main existence theorems, Theorems 2.1, 2.2 and 2.3. Theorem 2.1 is a theorem in the small, i.e., it asserts the existence of a unique solution of (1.1) for sufficiently small r_1, r_2. Theorem 2.2 is an existence theorem in the large, i.e., it asserts the existence of a unique solution for arbitrary r_1 and r_2, under the additional assumptions, namely, that either $\Gamma_1(x)$ or $\Gamma_2(y)$ is constant and the point x^*, y^* is a point of intersection of $\Gamma_1(x)$ and $\Gamma_2(y)$.

Theorem 2.3 gives sufficient conditions for the existence in the large of a solution of (1.1), when $K_i = 0$, $i = 1, 2, 3, 4$. In §3, we apply the results of §2 to the boundary value problems

$$\begin{aligned} L(u) &= u_{xy} + au_x + bu_y + cu = d \quad \text{in} \quad R, \\ u_x(x, y) &= \alpha_0(x)u(x, y) + \alpha_1(x)u_y(x, y) + \sigma(x), \quad \text{on} \quad y = \Gamma_1(x) \\ u_y(x, y) &= \beta_0(y)u(x, y) + \beta_1(y)u_x(x, y) + \tau(y), \quad \text{on} \quad x = \Gamma_2(y), \\ u(x^*, y^*) &= \gamma \end{aligned} \tag{1.2}$$

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