A QUESTION OF M. NEWMAN AND J. R. SMART

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Let Γ be the group of integral $t \times t$ matrices with determinant 1 and, if d divides n, let $M(d, n)$ be the group of matrices of form $I + dA$ in Γ, considered modulo n. In [1], M. Newman and J. R. Smart investigate the groups $M(d, n)$ and they reduce the determination of the structure of such groups to the case $M(p^\alpha, p^\beta)$ where p is a prime. For given t, they give a sufficient condition for an isomorphism between $M(p^\alpha, p^\beta)$ and $M(p^\delta, p^\epsilon)$, where $\alpha \geq 1$ and $\delta \geq 1$. They ask if there are any other cases for which the groups are isomorphic. We answer this question by showing that their condition is necessary as well as sufficient. Thus our main theorem states:

Theorem. *For $\alpha \geq 1$ and $\delta \geq 1$, the groups $M(p^\alpha, p^\beta)$ and $M(p^\delta, p^\epsilon)$ are isomorphic if and only if $\beta - \alpha = \epsilon - \delta$ and $\beta \leq 2\alpha$, $\epsilon \leq 2\delta$.*

Finally, for $t > 2$, we calculate the nilpotency class of the group $M(p^\alpha, p^\beta)$, where $\alpha \geq 1$.

Theorem 5 of [1] states:

Lemma 1. *If $1 \leq \alpha \leq \beta \leq 2\alpha$, then $M(p^\alpha, p^\beta)$ is abelian of order $p^{(\beta-\alpha)(t^2-1)}$.*

Newman and Smart then show:

Theorem 1. *If $1 \leq \alpha \leq \beta \leq 2\alpha$, $1 \leq \delta \leq \epsilon \leq 2\delta$, and $\beta - \alpha = \epsilon - \delta$, then $M(p^\alpha, p^\beta)$ and $M(p^\delta, p^\epsilon)$ are isomorphic.*

To begin our proof that these conditions are necessary, when $\alpha \geq 1$ and $\delta \geq 1$, we note:

Theorem 2. *If $0 \leq \alpha \leq \gamma \leq \beta$, then $M(p^\gamma, p^\beta)$ is a normal subgroup of $M(p^\alpha, p^\beta)$, and $M(p^\alpha, p^\beta)$ is an extension of $M(p^\gamma, p^\beta)$ by $M(p^\alpha, p^\gamma)$.*

Proof. $M(p^\gamma, p^\beta)$ is the kernel of the homomorphism from $M(p^\alpha, p^\beta)$ to $M(p^\alpha, p^\gamma)$ obtained by reducing the elements modulo p^γ.

Hence, if $|G|$ denotes the order of a group G, $|M(p^\alpha, p^\beta)| = |M(p^\alpha, p^{2\alpha})|$ $|M(p^{2\alpha}, p^\beta)| = p^{\alpha(t^2-1)}| M(p^{2\alpha}, p^{3\alpha})|$ $|M(p^{3\alpha}, p^\beta)| = \cdots = p^{(\beta-\alpha)(t^2-1)}$ and so certainly $\beta - \alpha = \epsilon - \delta$ is necessary for $M(p^\alpha, p^\beta)$ and $M(p^\delta, p^\epsilon)$ to be isomorphic, where $1 \leq \alpha \leq \beta$ and $1 \leq \delta \leq \epsilon$.

From now on we assume $\beta > 2\alpha$ and by calculating the order of the centre Z of $M(p^\alpha, p^\beta)$, we show that $M(p^\delta, p^\epsilon)$ is isomorphic to $M(p^\alpha, p^\beta)$ only if $\delta = \alpha$ and $\epsilon = \beta$. We shall need the following lemma from [2; 374].

Lemma 2. *If S is an integral $t \times t$ matrix with determinant congruent to 1 modulo m, there is an integral $t \times t$ matrix with determinant 1 which is congruent to S modulo m.*

Received May 8, 1964.