THE DIMENSION OF MAXIMAL COMMUTATIVE SUBALGEBRAS OF K_n

By R. C. Courter

1. Introduction. Throughout this paper K will denote a field and K_n will be the algebra of n by n matrices over K. The object of the paper is to refute the conjecture (referred to in [3; 345] as one of long standing) that a maximal commutative subalgebra of K_n must have dimension at least n. The counter-example, a 13-dimensional maximal commutative subalgebra of K_{14}, is presented in §3. In §4 we have a consequence of the example: Theorem 4.1, which states that zero is the greatest lower bound of the set of numbers $\{(\dim R)/n \mid R$ a maximal commutative subalgebra of K_n, $n = 1, 2, \cdots\}$. The Theorems prerequisite to these results are in §2.

Definition A. If P is the radical of the commutative algebra R, the exponent e of R is defined to be the index of nilpotency of P. Thus $P^e = 0$, $P^{e-1} \neq 0$.

The 13-dimensional maximal commutative subalgebra of K_{14} occurs at exponent 3. In §5 (Theorem 5.1) it is proved that the dimension of R is at least n, if the exponent of the maximal commutative subalgebra R of K_n is one or two.

It will be recalled that $1 + [n^2/4]$ is the maximal dimension of commutative subalgebras of K_n. This was proved by Schur [6] for the complex field K and by Jacobson [5] for arbitrary fields K.

2. Theorems prerequisite to the main results.

Definition B. If M is a unital R-module, where R is a commutative ring with unit element, we denote by $R^*(M)$, or simply by R^*, the set of R-endomorphisms of M, each of which is a right multiplication

$$a_r : x \longrightarrow xa, \qquad x \in M$$

for some element a of R.

Remark. If M is the n-dimensional representation space of the commutative subalgebra R of K_n, it is evident that the maximal commutativity of R in K_n is equivalent with the property

$$\mathrm{Hom}_R(M, M) = R^* .$$

Theorem 2.1. *If M is a cyclic unital R-module for the commutative ring*

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