THE NUMBER OF REPRESENTATIONS OF AN INTEGER AS A SUM OF TWO SQUARE-FREE NUMBERS

By Eckford Cohen

1. Introduction. Let n denote a positive integer, Q the set of square-free integers, and $T(n)$ the number of ordered pairs $\{a, b\}$ of elements in Q such that $n = a + b$. More than thirty years ago, Estermann proved [2] that

$$T(n) = cn\rho(n) + O(n^{2/3+\epsilon}) \tag{1}$$

for all $\epsilon > 0$, where

$$c = \prod_p \left(1 - \frac{2}{p^2}\right), \qquad \rho(n) = \prod_{p^2 \mid n} \left(1 + \frac{1}{p^2 - 2}\right), \tag{2}$$

it being understood that p stands for prime numbers.

Estermann's proof of (1) was entirely elementary. An elementary proof of Estermann's result of a fairly recent date is due to M. A. Subhankulov and S. N. Muhtarov (see Remark 1 below). In this paper we propose to prove (1) in the sharper form,

$$T(n) = cn\rho(n) + O(n^{2/3} \log^2 n), \qquad n \geq 2. \tag{3}$$

As far as the writer can determine, this is the first refinement of Estermann's result that has been proved. ((*Author's note.*) In the paper as originally submitted, (3) was proved with remainder $O(n^{2/3} \log^3 n)$. The improvement resulted from simplifications that the author introduced when this material was presented to a seminar in November, 1964, at the University of Tennessee.)

The method is elementary and is based upon a uniform approximation for the number of square-free integers not exceeding a given bound and contained in certain classes of residues. More precisely, for $x \geq 1$ and for integers a, r, with $r > 0$, let $Q_{a,r}(x)$ denote the number of $n \leq x$ such that $n \equiv a \pmod{r}$. Our proof is based on the following formula,

$$Q_{a,h^2}(x) = q((a, h^2))\left\{\frac{6x}{\pi^2}\left(\frac{1}{J(h)}\right) + O(x^{1/2}h^{-1/2}) + O(\theta(h)\sigma(h))\right\}, \tag{4}$$

uniformly in x, a, $h \geq 1$, where $J(h)$ is the Jordan totient of order 2,

$$J(h) = h^2 \prod_{p \mid h} \left(1 - \frac{1}{p^2}\right), \tag{5}$$

q is the characteristic function of Q, $\sigma(r)$ is the sum of the divisors of r, and $\theta(r)$ is the number of square-free divisors of r. For the later discussion we note the

Received October 13, 1963.