# AN INTEGRAL THEOREM FOR ANALYTIC INTRINSIC FUNCTIONS ON QUATERNIONS

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**1. Introduction.** $\mathfrak{Q}$, the algebra of real quarternions, is a four-dimensional division algebra over the real field $\mathfrak{R}$ with basis $i_0 = 1, i_1, i_2, i_3 = i_1 i_2$ and with multiplication determined by the associative and distributive laws and $i_1 i_2 = -i_2 i_1, i_1^2 = i_2^2 = -1$. A typical element of $\mathfrak{Q}$ is $\xi = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$ with the $x_i$ from the real field $\mathfrak{R}$. We will find it convenient to write $\xi$ in the form $\xi = x_0 + p\mu$ where $x_0$ and $p = \sqrt{x_1^2 + x_2^2 + x_3^2}$ are real and where $\mu = (x_1 i_1 + x_2 i_2 + x_3 i_3)/p$ is a unit vector quaternion, i.e. $\mu^2 = -1$. For fixed $\mu$, the elements of the form $x_0 + p\mu$ form a subspace of $\mathfrak{Q}$ isomorphic to the complex field $\mathfrak{C}$. Such a subspace will be called a complex plane, or a complex field, of $\mathfrak{Q}$ and will be denoted by $\mathfrak{C}(\mu)$.

Since $\mathfrak{Q}$ is a generalization of the algebra of complex numbers it is only reasonable that there have been several attempts to find a distinguished class of functions on $\mathfrak{Q}$ having properties analogous to the class of analytic functions of a complex variable. R. Fueter and his students (for a complete list of references see [3]) had considerable success in this direction, using, as a generalization of the Cauchy–Riemann equations, the left-regularity condition,

$$\left[\frac{\partial}{\partial x_0} + i_1 \frac{\partial}{\partial x_1} + i_2 \frac{\partial}{\partial x_2} + i_3 \frac{\partial}{\partial x_3}\right] F(\xi) = 0. \tag{1.1}$$

From this, and the corresponding right-regularity condition, they developed analogies of Cauchy's Theorem, Taylor's Theorem, and Morera's Theorem and many other results.

The properties of the regular functions are certainly very desirable, but it is somewhat disappointing that this class does not even contain the identity function $f(\xi) = \xi$, or any other polynomial in $\xi$.

Nisigaki [5] attempted to obtain an analogy of Cauchy's Theorem for functions whose Hausdorff-differential had three or less terms. Nisigaki's results are incorrect as is shown in [1].

Rinehart [8] has introduced and motivated the study of the class of intrinsic functions on a linear associative algebra $\mathfrak{A}$, with identity, over a field $\mathfrak{F}$. Let $G$ be the group of all automorphisms and antiautomorphisms of $\mathfrak{A}$ which leave $\mathfrak{F}$ element-wise invariant.

Definition 1.1. *A set of elements $\mathfrak{D}$ will be called an intrinsic set of $\mathfrak{A}$ if $\Omega\mathfrak{D} = \mathfrak{D}$ for every $\Omega$ in $G$.*

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