NORMAL ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KINDS

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1. Introduction. Legendre's choice of normal elliptic integrals is ordinarily adopted in practical work, particularly in integral tables, numerical tables, and computer programs. Weierstrass' notation is seldom used in these contexts, in part because it is designed primarily for elliptic functions rather than integrals, in part because Legendre's integrals were first to be extensively tabulated, and perhaps because it is sometimes bothersome to express an integral of the second kind in terms of the Weierstrass zeta function. Unfortunately, certain transformation formulas that are symmetrical or even trivial in Weierstrass' notation are fairly clumsy in Legendre's. This is by no means a purely academic matter, for the linear transformations are used to put the modulus and amplitude within ranges that have been tabulated, and the quadratic transformations are used for direct calculation.

In this paper we describe a choice of normal elliptic integrals which appears to be very well suited for use in integral tables as well as for numerical work. The associated transformation formulas are simple, and the hypergeometric character of elliptic integrals is brought to the forefront. After a brief review of the hypergeometric R function in §2, special cases of this function are chosen in §3 to be the complete integrals $R_K(z_1, z_2)$ and $R_E(z_1, z_2)$ and the incomplete integrals $R_F(z_1, z_2, z_3)$ and $R_G(z_1, z_2, z_3)$. All four functions are homogeneous and symmetric in their arguments, and the symmetry is shown in §4 to be the counterpart of the linear transformations of Legendre's integrals. The quadratic Landen and Gauss transformations take a unified form, derived in §5 as a special case of a new quadratic transformation of the R function. In §6 a type of practical problem is considered for which the present choice of normal integrals offers a special advantage: geometrical and physical properties of ellipses and ellipsoids (for example, the surface area of an ellipsoid) can be represented without obscuring their symmetry in the semi-axes. Two definite integrals that are useful in this connection are evaluated in terms of the R function.

Normal elliptic integrals of the third kind can be defined in a similar way but are not considered in the present paper.

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