FUNCTIONS ON ALGEBRAS UNDER HOMOMORPHIC MAPPINGS

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Introduction. In this paper an algebra will denote a linear, associative, finite dimensional algebra, with identity, over the real or complex field $\mathfrak{F}$. Let $\mathfrak{A}$ be an algebra and let the algebra $\mathfrak{B}$ be the image of $\mathfrak{A}$ under a homomorphism or anti-homomorphism Ω. A function f with domain and range in $\mathfrak{A}$ does not usually induce a (single-valued) function in $\mathfrak{B}$ under Ω, i.e. for ξ in the domain of f, the induced "function" g in $\mathfrak{B}$ defined by $g(\Omega\xi) = \Omega f(\xi)$ need not satisfy: $\Omega\xi_1 = \Omega\xi_2$ implies $g(\Omega\xi_1) = g(\Omega\xi_2)$.

The present paper seeks conditions on the function f under which Ω does induce a function in $\mathfrak{B}$. It is found that mere Hausdorff-differentiability of f will insure this, and that the induced function in $\mathfrak{B}$ will also be Hausdorff-differentiable. It is also shown that any primary function [5] (i.e. the extension to $\mathfrak{A}$ of a function $w(z)$ of a complex variable) has this property, whether Hausdorff-differentiable or not, and that the function induced in $\mathfrak{B}$ will be the $\mathfrak{B}$-extension of the same complex function $w(z)$.

2. **Congruence-preserving functions.** It is natural to first examine the behavior of the function f with domain and range in $\mathfrak{A}$ with respect to the ideals of $\mathfrak{A}$.

DEFINITION 2.1 *A function f with domain $\mathfrak{D}$ and range in the algebra $\mathfrak{A}$ is congruence-preserving in $\mathfrak{D}$, if for any ideal $\mathfrak{J}$ of $\mathfrak{A}$, ξ, $\eta \in \mathfrak{D}$ and $\xi \equiv \eta(\mathfrak{J})$ imply $f(\xi) \equiv f(\eta)(\mathfrak{J})$.*

THEOREM 2.1. *If a function f with domain and range in $\mathfrak{A}$ is H-differentiable on an open convex set $\mathfrak{C}$ of $\mathfrak{A}$, then f is congruence-preserving in $\mathfrak{C}$.*

Let us first recall the definition of H(ausdorff)-differentiability [2, 7]. Let $\epsilon_1, \cdots, \epsilon_n$ be a basis for $\mathfrak{A}$ with ϵ_1 the identity of $\mathfrak{A}$.

DEFINITION 2.2 *A function $f = \sum_{i=1}^n f_i(x_1, \cdots, x_n)\epsilon_i$ with domain $\mathfrak{D}$ and range in an algebra $\mathfrak{A}$ is Hausdorff differentiable at a point $\xi = \sum_{i=1}^n x_i\epsilon_i$ of an an open set $\mathfrak{R}$ in $\mathfrak{D}$ if the differential*

$$d(f(\xi); d\xi) = \sum_{i,j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_j\epsilon_i,$$

($d\xi = \sum_{i=1}^n dx_i\epsilon_i$) exists for all $d\xi \in \mathfrak{A}$ and is expressible in the form

$$d(f(\xi); d\xi) = \sum_{i,j=1}^{n} P_{ij}\epsilon_i \, d\xi\epsilon_j \tag{2.1}$$

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