SOME RADIUS OF CONVEXITY PROBLEMS

By Richard J. Libera

0. **Introduction.** Let $\mathcal{S}$ be the class of functions $f(z)$ which are regular and univalent (schlicht) in the open disk $|z| < 1$ (hereafter called E) and which are normalized by the conditions

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1; \tag{0.1}$$

and denote by $\mathcal{P}$ the functions $P(z)$ which are regular in E and satisfy the conditions

$$P(0) = 1 \quad \text{and} \quad \operatorname{Re}\{P(z)\} > 0, \qquad \text{for } z \text{ in } E. \tag{0.2}$$

It is well-known [5], [9] that a function with derivative of positive real part in a convex domain in univalent there, consequently, the class $\mathcal{R}$ of functions whose first derivatives are in $\mathcal{P}$ is a subset of $\mathcal{S}$, that is $\mathcal{R} \subset \mathcal{S}$. This theorem has been at the origin of several investigations [1], [2] and is the basis for the definition of "close-to-convex" regular functions introduced by W. Kaplan [3].

It is the purpose of this paper to determine the radius of convexity of $\mathcal{R}$, then to apply the results of that investigation to the determination of the radii of convexity and other mapping properties of some classes of close-to-convex functions. All functions considered have representations in terms of members of $\mathcal{P}$.

1. **Definitions and preliminaries.** If $\mathcal{A}$ is an arbitrary subclass of $\mathcal{S}$, then r. c. $\mathcal{A}$, the radius of convexity of $\mathcal{A}$, is the largest value r, $0 < r \leq 1$, for which

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq 0,$$

for all $|z| \leq r$ and all $f(z)$ in $\mathcal{A}$. r.c. $\mathcal{R}$ can be found by solving the problem

$$\underset{\substack{|z|=r \\ P(z)\in\mathcal{P}}}{\text{minimum}} \operatorname{Re}\left\{\frac{zP'(z)}{P(z)}\right\}, \qquad 0 < r \leq 1. \tag{1.1}$$

In a recent paper [7], M. S. Robertson developed a variational method for functions of $\mathcal{P}$; a solution to (1.1) is obtained here by an application of this technique. If $P(z)$ is in $\mathcal{P}$, then [7] so is

$$P^*(z) = P(z) - \rho^2(1 - |z_0|^2)zS(z) + o(\rho^2), \tag{1.2}$$

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