LAGUERRE TRANSFORMS

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1. **Introduction.** If $L_n(x)$ is the Laguerre polynomial of degree n,

$$L_n(x) = \frac{1}{n!} e^x \left(\frac{d}{dx}\right)^n [e^{-x} x^n] \qquad n = 0, 1, \cdots, \tag{1}$$

then

$$\int_0^\infty L_n(x) L_m(x) e^{-x}\, dx = \delta(n, m) \tag{2}$$

where $\delta(n, m)$ is 0 or 1 as $n \neq m$ or $n = m$. We have the recursion formula

$$-(n + 1)L_{n+1}(x) + (2n + 1)L_n(x) - nL_{n-1}(x) = xL_n(x) \tag{3}$$

which is valid for $n = 0, 1, \cdots$. ($L_{-1}(x)$ is to be interpreted as 0.)

It was shown in [2] that if

$$E(x) = e^{cx} \prod_k \left(1 + \frac{x}{a_k}\right) \tag{4}$$

where

$$0 \leq c, \qquad 0 < a_k, \qquad \sum_k a_k^{-1} < \infty, \tag{5}$$

and if

$$G(n, m) = \int_0^\infty [E(x)]^{-1} L_n(x) L_m(x) e^{-x}\, dx, \tag{6}$$

then the infinite matrix $[G(n, m)]$ $n, m = 0, 1, \cdots$ is totally non-negative; that is, if $0 \leq m_1 < m_2 < \cdots < m_r$, $0 \leq n_1 < n_2 < \cdots < n_r$, then

$$\det \begin{bmatrix} G(n_1, m_1) & G(n_1, m_2) & \cdots & G(n_1, m_r) \\ G(n_2, m_1) & G(n_2, m_2) & \cdots & G(n_2, m_r) \\ \vdots & \vdots & & \\ G(n_r, m_1) & G(n_r, m_2) & \cdots & G(n_r, m_r) \end{bmatrix} \geq 0. \tag{7}$$

Received August 6, 1962. This research was supported in part by the United States Air Force through the Air Force Office of Scientific Research and Development Command under Contract No. AF49(638)-846.