ABSTRACT MEAN VALUES

By Trevor Evans

1. **Introduction.** The entropic law for a binary operation $+$ on a set S, that is,

$$(x + y) + (z + w) = (x + z) + (y + w) \quad \text{for all} \quad x, y, z, w \quad \text{in} \quad S,$$

has been studied by many authors (see [13] and the references given there). In particular, this law or its n-ary generalization is a natural condition to assume in characterizing mean value functions on the real numbers, [1], [2]. This paper is a study of the n-ary generalization of the entropic law and has two main objects (i) to describe the structure of an n-ary operation on an abstract set satisfying the law, (ii) to give algebraic versions of some of the characterizations of mean value functions on the real numbers which have been discussed by various authors.

In [9] and [12], Kolmogorov and Nagumo have shown that if M_n, $n = 1, 2, 3, \cdots$ is an infinite sequence of strictly increasing continuous symmetric idempotent functions on the real numbers such that for all $k < n$, and all $x_1, x_2, \cdots, x_n$,

$$M_n(x_1, x_2, \cdots, x_n) = M_n(M_k, M_k, \cdots, M_k, x_{k+1}, \cdots, x_n),$$

(where M_k denotes $M_k(x_1, x_2, \cdots, x_k)$), then each M_n is a generalized arithmetic mean.

We prove that if the same algebraic conditions hold for an infinite sequence of operations $M_1, M_2, \cdots, M_n, \cdots$ on a set S with the continuity and order conditions replaced by the assumption that S contains an element g such that the mapping $x \rightarrow M_n(x, g, g, \cdots, g)$ is one-one onto S, for each n, then each M_n is an arithmetic mean on a certain commutative semigroup $(S, +)$.

We obtain the above result as a consequence of the algebraic analogue of the following result due to Aczél [1]. If M is a continuous, strictly increasing idempotent function of n variables on the reals such that, for any $n \times n$ matrix of real numbers, M satisfies the generalized entropic law

$$M\{M(\mathbf{r}_1), M(\mathbf{r}_2) \cdots, M(\mathbf{r}_n)\} = M\{M(\mathbf{c}_1), M(\mathbf{c}_2), \cdots, M(\mathbf{c}_n)\}$$

where $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n, \mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n$ are the row and column vectors of the matrix, then M is a generalized weighted arithmetic mean.

We prove that if an n-ary operation M on a set S satisfies the algebraic conditions assumed by Aczél and if S contains an element g satisfying the regularity condition that at least two of the mappings

$$x \rightarrow M(x, g, g, \cdots, g), x \rightarrow M(g, x, g, \cdots, g), \cdots, x \rightarrow M(g, g, \cdots, g, x)$$

Received March 3, 1962.