THE n-th DERIVATIVE OF A CLASS OF FUNCTIONS

By David Zeitlin

1. Recently, Carlitz [1], using Lagrange's formula (see [4; 125]), has shown, for arbitrary r, that

$$\left[\frac{d^n}{dx^n}\left(\frac{x-1}{x^r-1}\right)^{n+1}\right]_{x=1} = (1-r)(1-2r)\cdots(1-nr)r^{-n-1}. \tag{1.1}$$

In this paper, we will establish formulas for the n-th derivative of a class of functions, which, as one application, yield (1.1) as a special case, as well as other identities involving Bernoulli and Stirling numbers.

2. The following theorem provides a generalization of (1.1):

Theorem 1. *For all real $r \neq 0$, and $m \geq n+1$, $n = 0, 1, \cdots,$*

$$\left[\frac{d^n}{dx^n}\left(\frac{x-1}{x^r-1}\right)^{m}\right]_{x=1} \tag{2.1}$$
$$= \frac{n!\,(-1)^n}{r} \sum_{s=m-n-1}^{m-1} (-1)^s \sum_{j=0}^{m-n-1} (-1)^j \binom{m-n-1}{j}\binom{(j+1)/r}{s};$$

for $m = n+1$, (2.1) reduces to (1.1).

Proof. By Lagrange's general formula [4, 125], if $w = x/\phi(x)$, $\phi(0) \neq 0$, then

$$\frac{f(x)}{1-w\phi'(x)} = \sum_{k=0}^{\infty} \frac{w^k}{k!}\left[\frac{d^k}{dx^k}\{f(x)(\phi(x))^k\}\right]_{x=0}. \tag{2.2}$$

If we choose $\phi(x) = x/[(x+1)^r - 1]$, $f(x) = [\phi(x)]^{m-n}$, then $w = (x+1)^r - 1$, and

$$\frac{f(x)}{1-w\phi'(x)} = \frac{1}{r}\left(\frac{x+1}{w+1}\right)\left(\frac{x}{w}\right)^{m-n-1} = \sum_{k=0}^{\infty}\frac{w^k}{k!}\left[\frac{d}{dx^k}\{\phi(x)\}^{m-n+k}\right]_{x=0}. \tag{2.3}$$

We now proceed to expand the left-hand side of (2.3) into a power series in w. Now

$$x+1 = (w+1)^{1/r} = \sum_{k=0}^{\infty}\binom{1/r}{k}w^k, \qquad |w| < 1, \quad \text{and}$$

$$\begin{aligned}(x+1)x^{m-n-1} &= (x+1)(x+1-1)^{m-n-1} \\ &= \sum_{j=0}^{m-n-1}(-1)^{m-n-1-j}\binom{m-n-1}{j}(x+1)^{j+1} \\ &= \sum_{k=0}^{\infty}\left[\sum_{j=0}^{m-n-1}(-1)^{m-n-1-j}\binom{m-n-1}{j}\binom{(j+1)/r}{k}\right]w^k.\end{aligned}$$

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