THE INTEGRATION OF THE GENERALIZED DERIVATIVES

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1. Introduction. We begin with

Definition 1.1. Let $F(x)$ be a single-valued function defined over a given domain. We define the operator H_n as follows

$$(1.1) \quad H_n[F(x): x_1, \cdots, x_{n+1}] = (x_{n+1} - x_1) \cdots (x_{n+1} - x_n) \sum_{j=1}^{n+1} F(x_j)/\{(z - x_1) \cdots (z - x_{n+1})\}'_{z=x_j}$$

for $n = 1, 2, 3, \cdots$, where the "prime" denotes ordinary differentiation.

If we set $w_{n+1}(z) = (z - x_1) \cdots (z - x_n)$, then formula (1.1) becomes

$$H_n[F(x): x_1, \cdots, x_{n+1}] = w_{n+1}(x_{n+1}) \sum_{j=1}^{n+1} F(x_j)/w'_{n+2}(x_j).$$

In order to obtain $H_n[F(x): u_1, \cdots, u_n, x]$ it suffices to set $x_1 = u_1, \cdots, x_n = u_n$, $x_{n+1} = x$ in formula (1.1).

The following properties of H_n are direct consequences of Definition 1.1:

$$\text{(i)} \quad H_n\left[\sum_{i=1}^{m} a_i F_i(x): x_1, \cdots, x_{n+1}\right] = \sum_{i=1}^{m} a_i H_n[F_i(x): x_1, \cdots, x_{n+1}]$$

where $a_1, \cdots, a_m$ are arbitrary constants.

(ii) If $F(x)$ reduces to a polynomial of degree at most $n - 1$, then

$$H_n[F(x): x_1, \cdots, x_{n+1}] = 0.$$

(iii) If $x_{n+1} = x_i$, $(i = 1, \cdots, n; n \geq 1)$, then

$$H_n[F(x): x_1, \cdots, x_{n+1}] = 0.$$

(iv) $H_n[F(x): x_1, \cdots, x_{n+1}]$ remains invariant under all permutations of the points $x_1, \cdots, x_n$.

$$\text{(v)} \quad H_n[\{w_{n+1}(x_{n+1})\}^{-1}F(x): x_1, \cdots, x_{n+1}]$$

remains invariant under all permutations of the distinct points $x_1, \cdots, x_{n+1}$.

By means of H_n we define a generalized derivative as follows

Definition 1.2. Let $f(x)$ be defined and continuous in the interval (a, b) and let $x_1, \cdots, x_{n+1}$ be distinct constants. Suppose that the points x, $x + x_1h, \cdots, x + x_{n+1}h$ belong to the interval (a, b). If

$$(1.2) \quad H_n[n!\, \{h^n w_{n+1}(x_{n+1})\}^{-1} f(x): x + x_1h, \cdots, x + x_{n+1}h]$$

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