UNITARY DILATIONS WHICH ARE ORTHOGONAL BILATERAL SHIFT OPERATORS

To the memory of Maurice Audin

By Israel Halperin

1. Introduction.

1.1. In this paper we are concerned with the following question. Suppose $\{U_\alpha\} = \{U_\alpha \mid \alpha \in J\}$ is a minimal unitary dilation of contractions $\{T_\alpha\}$ on a Hilbert space H; what conditions on $\{T_\alpha\}$ force $\{U_\alpha\}$ to be bilateral shift operators?

1.2. We recall some definitions (see [4]). J denotes a totally ordered set of indices α and $\tilde{J}$ denotes the set of those integer valued functions $m \equiv m(\alpha)(-\infty < m(\alpha) < \infty)$ for which $\tilde{m} \equiv \{\alpha \mid m(\alpha) \neq 0\}$ is a finite subset of J.

We write $m = 0$, $m \geq n$ if for all α, $m(\alpha) = 0$, $m(\alpha) \geq n(\alpha)$ respectively. We define $m - n$ by $(m - n)\alpha = m(\alpha) - n(\alpha)$ for all α, and we write $-m$ for $0 - m$. We call *m, n positive-disjoint* if $m \geq 0$, $n \geq 0$ and for each α at least one of $m(\alpha)$, $n(\alpha)$ is 0.

If T_α, $\alpha \in J$ are bounded linear operators on a Hilbert space and $m \geq 0$, we set $T(m) = T_1^{m(1)} \cdots T_r^{m(r)}$ where the indices in $\tilde{m}$, ordered as in J, have been denoted $\{1, \cdots, r\}$ for convenience; we define $T(-m)$ to be $(T(m))^*$. By convention, $T(0) = 1$.

If U_α, $\alpha \in J$ are commuting unitary operators on a Hilbert space K, and $m \in \tilde{J}$ (we do not require $m \geq 0$), then $U(m)$ shall mean $U_1^{m(1)} \cdots U_r^{m(r)}$ where $\tilde{m} = \{1, \cdots, r\}$. We shall write $U(A)$ for the subspace spanned by $\{U(m)A \mid m \in \tilde{J}\}$ for a given subspace A of K. $\{U_\alpha\}$ are said to be *orthogonal bilateral shift operators on K with S as shifted space* if S is a subspace of K and $\{U(m)S \mid m \in \tilde{J}\}$ are mutually orthogonal and span K (that is, $U(S) = K$).

Commuting unitary operators $\{U_\alpha\}$ acting on a Hilbert space $K \supset H$ are called a *unitary dilation* of $\{T_\alpha\}$ acting on H if for all $x \in H$:

$$T(m)x = P_H U(m)x \quad \textit{for} \quad m \geq 0. \tag{1.1}$$

Here P_H denotes the projection (orthogonal) onto H.

Note. $\{U_\alpha\}$ are required to be commuting but not $\{T_\alpha\}$.

The dilation $\{U_\alpha\}$ is called a *Sz.-Nagy-Brehmer dilation* if:

(1.2) $T(-n)T(m)x = P_H U(-n)U(m)x$ *for positive-disjoint* m, n *and* $x \in H$,

equivalently,

$(U(m)x \mid U(n)y) = (T(m)x \mid T(n)y)$ *for positive-disjoint* m, n *and* $x \in H$, $y \in H$;

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