THE GROWTH OF SYLVESTER'S CYCLOTOMIC NUMBERS

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1. Introduction. If L and M are rational integers, $L > 0$, $M \neq 0$, the sequence (P) defined by

$$
(1)\qquad \begin{aligned} &P_0 = 0, \quad P_1 = 1, \quad P_2 = 1, \quad P_3 = L - M, \\ &P_n = (L - 2M)P_{n-2} - M^2 P_{n-4}, \quad \text{if } n \geq 4, \end{aligned}
$$

is called a *Lehmer sequence*. If L is a square, the sequence (U) given by

$$U_{2n} = L^{\frac{1}{2}} P_{2n}, \qquad U_{2n+1} = P_{2n+1}$$

is called a *Lucas sequence*. Here $U_0 = 0$, $U_1 = 1$, and $U_n = L^{\frac{1}{2}} U_{n-1} - MU_{n-2}$, for $n \geq 2$. It is clear that the terms of these sequences are rational integers. Moreover, (P) and (U) are *divisibility sequences*, since $n \mid m$ implies $P_n \mid P_m$ and $U_n \mid U_m$.

If (S) is a Lehmer sequence, a Lucas sequence, or a Sylvester sequence (defined below), an index n is called an *exceptional* index of (S) if $n > 2$ and if each prime dividing the term S_n with index n divides a term with smaller positive index. Any given Lucas sequence or Lehmer sequence has only finitely many exceptional indices, if any, when $L > 4M$. (This case is called the "real" case. Cf. Lekkerkerker [4], Durst [3].) The case $L = 4M$ is trivial: n is exceptional unless it is a prime not dividing M. For $L < 4M$ (the "complex" case), many exceptional indices of individual sequences are known, but it is not known if there is a complex sequence with no exceptional indices, or if there are any non-degenerate complex sequences with infinitely many exceptional indices. (A sequence (P) is degenerate if $P_n = 0$ for some positive n.)

In [3], Lekkerkerker's theorem was shown to be a simple consequence of results of Ward [5] and Durst [2]. The purpose of this paper is to accomplish a further reduction and simplification. Although Ward's argument is technically "elementary", his result can be obtained by methods more direct and far less sophisticated than those he used. The simpler proof given here is based on a device of Carmichael (§4) and a property of Fibonacci's sequence, apparently previously unnoted, which is of some interest in its own right (§3).

The method presented here for the real case $L > 4M$ cannot be extended in any very obvious way to the complex case $L < 4M$, which remains mysterious.

2. Sylvester sequences. The Lehmer sequence (1) is said to be *generated* by the polynomial $z^2 - L^{\frac{1}{2}}z + M$. Indeed, if $K = L - 4M \neq 0$ and α, β are the roots of its generator, the terms of (1) are given by

$$P_{2n} = (\alpha^{2n} - \beta^{2n})/(\alpha^2 - \beta^2), \qquad P_{2n+1} = (\alpha^{2n+1} - \beta^{2n+1})/(\alpha - \beta).$$

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