RAMIFIED COVERINGS OF RIEMANN SURFACES

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Here we continue the work started by A. Hurwitz [2] on ramified coverings. He introduced the general idea of describing a ramified covering by its associated unramified covering, and he also investigated the case of coverings of simply connected surfaces. We begin by developing a precise setting for the general classification of ramified coverings. Then we apply this to the determination of all finite ramified coverings of compact surfaces and bordered surfaces of genus strictly greater than zero. The Hurwitz relation for the genera of a covering of a compact surface is generalized to bordered surfaces.

1. **Preliminaries on unramified coverings.** We begin with some results on unramified coverings of spaces (i.e., topological spaces). These spaces are assumed to be locally simply connected and locally arcwise connected. For example, these spaces can be manifolds or manifolds with boundary.

Let X be a connected (therefore, arcwise connected) space. Let (Y, π, X) be an unramified covering where $\pi : Y \rightarrow X$ is a local homeomorphism and where for all paths u in X and all $y_0 \in Y$, $\pi(y_0) = u(0)$, there exists a path v in Y such that $\pi \circ v = u$ and $v(0) = y_0$. Y is not necessarily connected. If (Y, π, X) and (Y', π', X) are two coverings of X, then we say that they are *isomorphic* if there is a map $f : Y \rightarrow Y'$ such that $\pi = \pi' \circ f$ and f is a bijection.

If (Y, π, X) is an unramified covering and if Y' is a connected component of Y, then $\pi \mid Y' : Y' \rightarrow X$ is an unramified covering. (Y, π, X) is a *finite covering* if for all $x \in X$, $\pi^{-1}(x)$ is a finite set. The number of elements in $\pi^{-1}(x)$ is independent of x and is called the *order* of the covering. A covering (Y, π, X) is of *finite type* if for each connected component Y' of Y, $\pi \mid Y' : Y' \rightarrow X$ is a finite covering.

Let $x_0 \in X$, let (Y, π, X) be a covering, and let $E = \pi^{-1}(x_0)$. Then $\pi_1(X, x_0)$ operates (on the right) on the set E. This is defined for $\{u\} \in \pi_1(X, x_0)$ and $y \in E$ by $y\{u\} = v(1)$ where v is the unique lift of u to Y such that $v(0) = y$. Note that $y1 = y$ and $y\{uu'\} = (y\{u\})\{u'\}$ where $1, \{u\}, \{u'\} \in \pi_1(X, x_0)$. Note that Y is connected if and only if $\pi_1(X, x_0)$ acts transitively on E. In general, the connected components Y_i of Y are determined by E_i, the transitivity sets (or orbits) of the action of $\pi_1(X, x_0)$ on E.

Let F and F' be two sets on which $\pi_1(X, x_0)$ operates on the right. F and F' are $\pi_1(X, x_0)$*-isomorphic* if and only if there exists a bijection $f : F \rightarrow F'$ such that $f(y\{u\}) = f(y)\{u\}$ for all $y \in F$ and $\{u\} \in \pi_1(X, x_0)$.

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