CONCERNING LOCAL SEPARABILITY IN METRIC SPACES

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In [1] F. B. Jones raised the following question: "Is every connected, locally peripherally separable, metric space separable?" In [3] the author answered this question by showing the existence of a non-separable complete metric space Σ satisfying the above conditions, and, furthermore, which has the property that the set W of all points at which Σ is not locally separable is a totally disconnected perfect set which is separable, but not compact. In [2] the author showed that if Σ' is any non-separable space satisfying the conditions stated in Jones' question, then the corresponding set W' is uncountable. It is the purpose of this paper to characterize W' still further by showing that the set W' is a non-compact perfect set. This is done by proving a general theorem which shows that certain non-connected metric spaces are separable, and then obtaining the previous statement as a corollary. An example is also given of such a Σ', where Σ' is complete and W' is a separable continuum.

Let S denote a topological space. S is said to be locally separable at the point P provided it is true that if P is a point of the open set U, then there is an open set V such that $P \varepsilon V \subset U$ and V is separable. S is said to be locally peripherally separable at the point P provided it is true that if P is a point of the open set U, then there is an open set V such that $P \varepsilon V \subset U$ and $\bar{V} - V$ is separable. S is locally peripherally separable provided S is locally peripherally separable at each point.

Suppose M is a closed subset of the metric space S such that (1) M is separable and $S - M$ is locally separable, and (2) there exists a sequence of open sets $D_1, D_2, D_3, \cdots$ closing down on M such that for each n, D_n contains $\bar{D}_{n+1}$ and has a separable boundary. Let S' be the metric space whose set of all points is $(S - M) + \{M\}$ and a basis for which is the collection G such that g belongs to G if and only if $g = (D_i - M) + \{M\}$ for some i, or g is an open subset of $S - M$.

Theorem. *If S' is connected, S is separable.*

Proof. S' can easily be shown to be metrizable by showing that the first 3 parts of Moore's axiom C ([3; 700]) hold. S' is also locally peripherally separable. If S' is not separable, then S' must fail to be locally separable at each of uncountably many points. But the only point where S' can fail to be locally separable is $\{M\}$. So S' is separable, which means that $S - M$ is separable. Thus S is separable, because it is the sum of two separable sets.

Corollary. *Suppose T is a non-separable, locally peripherally separable,*

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