OPERATIONAL FORMULAS CONNECTED WITH TWO GENERALIZATIONS OF HERMITE POLYNOMIALS

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1. **Introduction.** The object of this paper is to develop certain operational formulas which arise from two forms of generalized Hermite polynomials. The resulting formulas allow a considerable unification of various special results which appear in the literature.

One of the customary ways to define the Hermite polynomials is by the relation

$$H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2}, \tag{1.1}$$

where $D = d/dx$.

Now, Burchnall [3] has employed the operational formula

$$(D - 2x)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H_{n-k}(x)\, D^k \tag{1.2}$$

to prove the formula of Nielsen [14]

$$H_{m+n}(x) = \sum_{k=0}^{\min(m,n)} (-2)^k \binom{m}{k} \binom{n}{k} k!\, H_{m-k}(x) H_{n-k}(x). \tag{1.3}$$

Recently Carlitz [7] has given an operational formula analogous to (1.2) but involving Laguerre polynomials. His formula is

$$\prod_{j=1}^{n} (x\, D - x + a + j) = n! \sum_{k=0}^{n} \frac{x^k}{k!} L_{n-k}^{(a+k)}(x)\, D^k, \tag{1.4}$$

where

$$L_n^{(a)}(x) = \frac{1}{n!} x^{-a} e^x D^n (x^{a+n} e^{-x}). \tag{1.5}$$

Now it is fairly easy to give examples of operational formulas similar to (1.4) but which do not involve Laguerre polynomials. Thus it is readily verified by induction that

$$\prod_{j=1}^{n} (x\, D + a + j) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+a}{n-k} (n-k)!\, x^k D^k. \tag{1.6}$$

These and many others are special instances of the more general formulas we give below.

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