BOUNDARY MEASURES OF ANALYTIC DIFFERENTIALS

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1. **Introduction.** We consider a compact set C in the complex plane with connected complement, and define a class H of analytic differentials on the interior U of C. In case C is the unit disk, for example, H will consist of all analytic differentials $f(z)dz$ with f in the class H_1 (see Zygmund [7]). We also consider the class M of finite complex-valued Borel measures on the boundary B of C which are orthogonal to polynomials. Our main result, Theorem 3 below, will establish a natural one to one correspondence between H and M. (Actually, Theorem 3 contains a somewhat more general result.) If $d\omega \in H$, the associated μ in M is roughly describable as the boundary measure of the analytic differential $d\omega$. The difficulty consists in showing that each μ in M is the boundary measure of *some* analytic differential $d\omega$ in H.

As a simple corollary of our main theorem, we shall derive Mergelyan's result (Theorem 4 below), that every continuous function on C which is analytic on U is uniformly approximable by polynomials.

This paper is a sequel to a previous paper [1]. Many of the results needed here were established in [1] in somewhat less generality. Since the proofs of certain of these results carry over word for word to the more general case considered here, we do not give them in such detail as would otherwise be necessary.

2. **The classes M and H.** We now introduce the sets with which we shall be working, which are somewhat more general than the sets with connected complement mentioned above.

Definition 1. The compact set C in the complex plane will be called balanced if the boundary B of C equals the boundary of the unbounded component of $-C$. The open set U in the complex plane will be called balanced if U is bounded, if the closure C of U is balanced, and if U is the interior of C.

The following lemma needs no proof.

Lemma 1. *A compact balanced set has a balanced interior. An open balanced set is simply connected. The union of any set of components of an open balanced set is an open balanced set.*

By measure we always mean finite complex-valued Borel measure. We now define the class M.

Definition 2. Let C be a compact balanced set in the complex plane. The class $M(C)$ is defined to be the class of all measures μ on the boundary B of C such that

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