A THEOREM ON THE MEASURABILITY OF GROUP-INVARIANT OPERATORS

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We give here a simple criterion for an invariant differential operator on a Lie group to be carried into a measurable operator in Hilbert space by the infinitesimal representation associated with a given unitary representation of the group. One type of application is to the treatment of a certain kind of mathematical model for elementary particles. In 'covariant' theories the 'quantum numbers' are for the most part operators of the type $dU(A)$ considered in the theorem. It follows that although the quantum numbers are generally unbounded operators, they nevertheless have unique spectral decompositions, and can be added and multiplied together freely, etc., for a fairly extensive class of such models. In more mathematical terms, one consequence is a theorem of Kostant dealing with the essential self-adjointness of operators on a semi-simple group that are invariant under a maximal essentially compact subgroup. Another is a previous result of ours similarly concerned with central operators on arbitrary Lie groups.

To define some of the technical terms and notations involved in the statement of our result, we recall that if $U(\cdot)$ is a continuous linear unitary representation on a (complex) Hilbert space $\underline{H}$ of a Lie group G, then there exists an infinitesimal representation $dU(\cdot)$ of the enveloping algebra $\underline{E}$ of (the Lie algebra $\underline{G}$ of) G, which is a *-homomorphism of $\underline{E}$ into an algebra of (unbounded) operators in $\underline{H}$, uniquely determined by the properties: (a) if X is in $\underline{G}$, then $idU(X)$ is extended by the infinitesimal self-adjoint generator of the one-parameter continuous unitary group $[U(\exp(tX)): -\infty < t < \infty]$; (b) the operators $dU(A)$ with $A \in \underline{E}$ all have the same domain $\underline{M}$, which they leave invariant, and which is maximal with respect to this property. As it is convenient in practice to employ a variety of domains, the (unique) domain just described may usefully be called the 'maximal' domain for $dU(\cdot)$. The * in $\underline{E}$ is definable as the unique adjunction operation taking X into $-X$, for all X in $\underline{G}$.

The notation $\underline{S}'$ for a set of continuous linear operators $\underline{S}$ indicates the collection of all continuous linear operators (on the same space) that commute with each element of $\underline{S}$. 'Ring of operators' is meant in the sense of Murray and von Neumann, i.e. as a weakly closed self-adjoint ring of continuous linear operators, among which is the identity. 'Measurable' operators, which are roughly the most general variety for which free manipulation is known to be legitimate, are treated in [6], where further references to the theory of rings of operators are given. The closure of an operator T, when it exists, is denoted as $\bar{T}$.

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