MEASURES INDUCED ON A σ-ALGEBRA BY A SURFACE

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1. Introduction. Let (T, A) be a c.BV mapping (continuous and of bounded variation; i.e., a continuous surface of finite area) from any admissible set $A \subset E_2$ into E_3, $p = T(w)$, $p = (x, y, z) \in E_3$, $w = (u, v) \in A$. (See [2] for all definitions.) For every point $p \in T(A)$ consider the components g of $T^{-1}(p) \subset A$. Let $\Gamma = \Gamma(A)$ be the collection of all such components $g \subset A$ as p varies over $T(A)$. The components are continua if A is compact. Let $\mathcal{B}_0$ be the class of all sets which are Borel measurable and are unions of components $g \in \Gamma$. Then $\mathcal{B}_0$ is a σ-algebra since A is Borel measurable. Let $\mathcal{G}_0$ be the class of all sets $G \in \mathcal{B}_0$ which are open in A. In particular every set $G \in \mathcal{G}_0$ is admissible. When A is compact, L. Cesari has defined, in [1] and [2], functions on $\mathcal{B}_0$ as follows:

$$\varphi(K) = \inf \{V(T, G) \quad : G \supset K, G \in \mathcal{G}_0\},$$

$$\varphi_r(K) = \inf \{V(T_r, G) \quad : G \supset K, G \in \mathcal{G}_0\},$$

$$\varphi_r^+(K) = \inf \{V^+(T_r, G) : G \supset K, G \in \mathcal{G}_0\},$$

$$\varphi_r^-(K) = \inf \{V^-(T_r, G) : G \supset K, G \in \mathcal{G}_0\},$$

for $r = 1, 2, 3$, where K is any set of $\mathcal{B}_0$. He has also defined $d_r(K) = \varphi_r^+(K) - \varphi_r^-(K)$. This will be denoted by $\mathcal{V}_r(K)$ in the present paper. We will also use these definitions even when A is not compact.

Among other results, the following theorem has been proved by Cesari in [1] and is given here in the terminology of P. Halmos [3]. This terminology will be used throughout this paper.

THEOREM 1. *Let (T, A) be a c.BV mapping from the compact admissible set A. Then the functions $\varphi, \varphi_r, \varphi_r^+, \varphi_r^-$ are regular measures on $(A, \mathcal{B}_0)$. The functions $\mathcal{V}_r$ are signed measures on $(A, \mathcal{B}_0)$. Moreover $\varphi(G) = V(T, G)$ for every set $G \in \mathcal{G}_0$ and similarly for $\varphi_r, \varphi_r^+, \varphi_r^-, V_r$.*

Let us note that $\varphi_r(K) = \varphi_r^+(K) + \varphi_r^-(K)$ for every $K \in \mathcal{B}_0$, under the hypotheses of Theorem 1. This may be seen by observing that if $\{G_n\}, \{M_n\}, \{N_n\}$ are sequences of sets in $\mathcal{G}_0$ all containing K such that $\varphi_r(K) = \lim V(T, G_n)$, $\varphi_r^+(K) = \lim V^+(T_r, M_n)$ $\varphi_r^-(K) = \lim V^-(T_r, N_n)$, then $\varphi_r(K) = \lim V(T_r, G_nM_nN_n) = \lim V^+(T_r, G_nM_nN_n) + \lim V^-(T_r, G_nM_nN_n) = \varphi_r^+(K) + \varphi_r^-(K)$, since $V(T_r, G) = V^+(T_r, G) + V^-(T_r, G)$ for every admissible set G.

Let (T, A) be any c.BV mapping from any admissible set A. Let $\Gamma(A)$ be as above. Let $\Gamma^0(A)$ be the class of all $g \in \Gamma(A)$ such that g is compact and $g \subset A^0$, where A^0 is the interior of A. Let $\hat{A}$ be the union of all $g \in \Gamma^0(A)$. The set $\hat{A}$ is open in E_2 (see [2; 221]). The purpose of the present paper is to prove:

Received April 26, 1958. This research was supported by OSR contract AF 18(600)-1484.