REPRESENTATIONS OF EVEN FUNCTIONS (mod r), III. SPECIAL TOPICS

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1. **Introduction.** This paper is the concluding article [4], [5] in a series of three papers concerning the theory of even functions (mod r). We shall assume the results, nomenclature, and notation of the first and second papers, denoted here by I and II, respectively. Reference numbers with an asterisk refer to the bibliography of I.

The paper is concerned with five topics: arithmetical averages, mean values, sums of finite primes, orthogonality properties, series expansions. We discuss briefly the content of the paper.

Section 2 is concerned with the average $A(f(n, r))$ of a real-valued even function $f(n, r)$,

$$A(f(n,r)) = \frac{1}{r} \sum_{a \pmod r} f(a,r). \tag{1.1}$$

By virtue of [9*, (1), (8)] one obtains

$$A(f(n,r)) = \alpha(r), \qquad (\alpha(r) = \alpha(1,r)), \tag{1.2}$$

where $\alpha(d, r)$, $d \mid r$, represents the general Fourier coefficient of $f(n, r)$. In the r-dimensional Euclidean space S_r of periodic functions (mod r), the even functions form a subspace E_r of dimension $\tau(r)$ = the number of divisors of r. Consequently, by Parseval's relation for finite dimensional spaces [8, §8.5, (7)] and the fact that E_r has the orthonormal basis, $c(n, d)/(r\phi(d))^{\frac{1}{2}}$, $d \mid r$, it follows that

$$A(f^2(n,r)) = \sum_{d \mid r} \alpha^2(d,r)\phi(d). \tag{1.3}$$

This relation is also a consequence of formulas (4.1) and (4.2) of II.

It is easily seen that $f(n, r)$ defines a discrete random variable (§2). This observation is used, in conjunction with results of elementary probability theory, to obtain in §2 estimates involving the averages of even functions (Theorems 1, 2 and Corollaries). A number of special functions discussed in I and II are used to illustrate the ideas of this section, for example, the functions $\theta(n, r)$, $\epsilon(n, r)$, $c^2(n, r)$, and $\tau(n, r)$.

In §3 an approximation to the mean value of $f(n, r)$ is obtained (Theorem 3). The proof of this result is based on a trigonometric estimate which sharpens an earlier estimate due to Carmichael (see Remark, §3).

Let J_r denote the residue class ring of the ring of integers (mod r). In §4 we are concerned with the function $G_s(\rho)$, defined to be the number of solutions

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