# REPRESENTATION THEOREMS AND INEQUALITIES FOR HERMITIAN MATRICES

By Richard Bellman

1. **Introduction.** In a series of papers, [1], [3], [4], it has been shown that the integral representation

$$\frac{\pi^{N/2}}{|A|^{\frac{1}{2}}} = \int_{-\infty}^{\infty} e^{-(x, Ax)}\, dx \tag{1.1}$$

where $A$ is a symmetric matrix whose real part is positive definite, $dx = dx_1 dx_2 \cdots dx_N$, and the integration is over the whole $x$-space, can be used to establish a number of inequalities derived in other ways; (cf. Ostrowski and Taussky [11], Ky Fan, [6], Oppenheim, [10].

On the other hand, the methods used by the authors of the papers cited above possess the advantage of being equally applicable to the study of Hermitian matrices, whereas (1.1), as it stands, can be used only for symmetric matrices.

In this paper we shall establish an analogue of (1.1) for positive definite Hermitian matrices, and then use this result to derive a number of known inequalities.

As a further example of the use of representation theorems of this type we shall use (1.1) and the Hermitian analogue to derive a partial generalization of a recent inequality of Hua, [7]. We shall then use a deeper representation theorem of Siegel, [12], and Ingham, [8], to obtain a further generalization. The result for Hermitian matrices requires a generalization of the Siegel result due to Braun, [5]. A generalization in a different direction, given in [2], enables still further results to be obtained.

In a recent paper by Marcus, [9], extensions of a still different nature are indicated.

2. **Evaluation of an integral.** By analogy with the formula of (1.1), let us discuss the integral

$$J(H) = \int_{-\infty}^{\infty} e^{-(\bar{z}, Hz)}\, dx\, dy, \tag{2.1}$$

where $z = x + iy$ and $H$ is a positive definite Hermitian matrix.

Write $H = A + iB$, $A$ and $B$ real, so that $A' = A$, $B' = -B$. It follows that

$$J(H) = \int_{-\infty}^{\infty} e^{-(x, Ax) - 2(Bx, y) - (y, Ay)}\, dx\, dy. \tag{2.2}$$

Since the integral is absolutely convergent, it may be evaluated by integration first over $x$ and then over $y$.

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