COEFFICIENT PROBLEMS FOR FUNCTIONS REGULAR IN AN ELLIPSE

By W. C. Royster

1. **Introduction.** Let $f(z)$ be an analytic function which is regular in an ellipse with foci at ± 1. It is known that such a function possesses an expansion in a series of Tchebychef polynomials in the largest ellipse in which $f(z)$ is regular [8]. We shall be concerned in this paper with the problem of obtaining bounds on the coefficients in the expansion whenever $f(z)$ belongs to certain classes of functions which have specified mapping properties. The classes which we shall consider are

(1) typically real functions (T)
(2) functions that are univalent and convex in the direction of the imaginary axis (F)
(3) functions starlike in the direction of the real axis (R)
(4) starlike functions (S)
(5) functions having a diametral line (D).

The class of typically real functions was first studied for Taylor series by Rogosinski [6] and later for Laurent series by Nehari and Schwarz [3]. The classes F and R were studied for Taylor series by Robertson [4], [5]. The class S has been studied for Taylor series by many authors and for Laurent series by Nehari and Schwarz [3], while the class D was studied by Umezawa [7] and De Bruijn [2].

2. **The class T.** The function $f(z)$ is said to be typically real in E if it is real when and only when z is real. The assumption that $f(z) \in T$ is equivalent to the assumption that $\operatorname{Im} f(z) \cdot \operatorname{Im} z$ possesses the same sign for any $z \in E$ for which $\operatorname{Im} z \neq 0$. Without loss of generality suppose $\operatorname{Im} f(z) \cdot \operatorname{Im} z > 0$. Since $f(z)$ is regular in E, it has an expansion in a series of Tchebychef polynomials

$$f(z) = \sum_{n=0}^{\infty} a_n T_n(z), \qquad z \in E, \tag{2.1}$$

where $T_n(z) = \cos n\,(\cos^{-1} z)$, which converges uniformly in E [8].

First let us relate the class of typically real functions in E to functions having positive real part in E. Let the parametric representation of E be given by $z = a_0 \cos t + i\, b_0 \sin t$, $0 \leq t < 2\pi$. Since $a_0^2 - b_0^2 = 1$, we let $a_0 = \cosh s_0$, $b_0 = \sinh s_0$, $s_0 > 0$. Any ellipse which is confocal with E and interior to E can be represented by $z = a \cos t + ib \sin t$, $0 \leq t < 2\pi$, with $a = \cosh s$, $b = \sinh s$, $0 < s \leq s_0$. Thus we may write $z = \cos (t - is)$.

Received May 5, 1958.