A CLASS OF INTEGRAL IDENTITIES WITH MATRIX ARGUMENT

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1. Introduction and summary. Let $X: p \times p$ be a real symmetric matrix, $\bar{X}^k = (x_{ij})$, $1 \le i, j \le k$, $\underline{X}^k = (x_{ij})$, $k \le i, j \le p$ and $A: p \times p$ a real positive definite matrix. Bellman [1] established the result

$$(1.1)\qquad \begin{aligned} \mathfrak{a}(a, k, A) &= \int_{X>0} \frac{|X|^{a_p-(p+1)/2} e^{-\operatorname{tr} AX}}{\prod_2^p |\underline{X}^i|^{k_{i-1}}}\, dX \\ &= \pi^{p(p-1)/4} \prod_1^p \Gamma[a_j - (j-1)/2]\, |\bar{A}^j|^{-k_j}, \end{aligned}$$

where $a_j = k_{p-j+1} + \cdots + k_p$, $k = (k_1, \cdots, k_p)$, $a = (a_1, \cdots, a_p)$. The integration is over the region where X is positive definite, and $a_j > (j-1)/2$, $j = 1, \cdots, p$. (Note a misprint in [1; 575]; read $p - R$ for R). (1.1) is a generalization of the identity of Ingham [3] and Siegel [5]. The proof of Bellman is inductive, and is similar to that of Ingham.

In this paper, an alternative and more elementary proof of (1.1) is presented (§2.) The method of proof is based on certain matrix transformations, and readily lends itself to the evaluation of a number of other integral identities. In §3, the integral over a certain set of positive definite correlation matrices is evaluated, and in §4, two extensions of Siegel's generalized Beta function are given. A matric analogue of the Liouville-Dirichlet integral is obtained in §5, which yields the Siegel-Ingham identity as an application. The integrals discussed are connected with distribution and moment problems in multivariate analysis, when the variables have a Wishart distribution. We also note that Bessel functions of matrix argument have been developed by Herz [2], which are extensions along other lines of some of the integrals considered.

The domain of integration $A < X < C$, (real symmetric matrices), denotes the set $\{X: C - X, X - A \text{ positive definite}\}$. $dX = \prod_{i<j} dx_{ij}$ if X is symmetric, $dX = \prod_{i,j} dx_{ij}$ if X is triangular, $dX = \prod_{i<j} dx_{ij}$ if X is a correlation matrix, i.e., $x_{ii} = 1$, $|x_{ij}| \le 1$, $i \ne j$, $i, j = 1, \cdots, p$. We will use the functions

$$\Gamma_p(a) = \pi^{p(p-1)/4}\Gamma(a)\Gamma(a - \tfrac{1}{2}) \cdots \Gamma\left(a - \frac{p-1}{2}\right),$$

$$\Gamma_p^*(a) = \Gamma_p^*(a_1 \cdots, a_p) = \pi^{p(p-1)/4}\Gamma(a_1)\Gamma(a_2 - \tfrac{1}{2}) \cdots \Gamma\left(a_p - \frac{p-1}{2}\right),$$

$$B_p(a, b) = \Gamma_p(a)\Gamma_p(b)/\Gamma_p(a + b),$$

$$B_p^*(a, b) = \Gamma_p^*(a)\Gamma_p^*(b)/\Gamma_p^*(a + b).$$

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