POLYNOMIAL COCYCLES

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1. **Introduction.** Let A be any additive group. Consider A as acting simply on itself; then according to the original formulation of cohomology of finite groups by S. Eilenberg and S. MacLane [2], one defines for every function f of n variables in A into A, a function δf, of $n + 1$ variables in A into A, by the formula

$$(1)\qquad \begin{aligned}(\delta f)(x_1, x_2, \cdots, x_{n+1}) = f(x_2, \cdots, x_{n+1}) \\ + \sum_{\nu=1}^{n} (-1)^{\nu} f(x_1, \cdots, x_\nu + x_{\nu+1}, x_{\nu+2}, \cdots, x_{n+1}) \\ + (-1)^{n+1} f(x_1, \cdots, x_n).\end{aligned}$$

A function f is called a *cocycle* if δf is 0, and a *coboundary* if $f = \delta g$ for some function g. If A is the additive group of a field, one may ask what cocycle-functions are expressible by polynomials. More precisely, let k be any field. For each $n > 1$, call a polynomial $f(x_1, \cdots, x_n)$ a *polynomial n-cocycle* if the polynomial $\delta f(x_1, \cdots, x_{n+1})$, defined by (1), is identically zero; call it a *polynomial n-coboundary* if it is identically $(\delta g)(x_1, \cdots, x_n)$ for some polynomial g, with cofficients in k, in $x_1, \cdots, x_{n-1}$. From now on we shall consider only polynomial cocycle and coboundaries, and shall omit the word "polynomial". Clearly every coboundary is a cocycle; we want to find whether there are any others.

Besides its interest as a problem of elementary algebra, this is worth studing for the following reasons: The 1-cocycles are precisely the additive polynomials, which have been shown to have an interesting theory [1], [5], [6]. The Witt-vector formalism [9] represents the additive group of a p-adic integral domain by a chain of group extensions by the additive group of its residue class field; the essential point is that the 2-cocycles describing these group extensions can be defined by polynomials. In generalized local class field theory it has been found that polynomial functions are of great importance [7], [8]. For application to Witt-vectors and to local class field theory we would like to solve the much harder problem of determining all polynomial cocycles over an integral domain rather than over a field; our present problem represents progress toward this harder one.

2. **Results.**

Definition. In any field, an *additive polynomial* is an $f(x)$ satisfying the identity $f(x + y) - f(x) - f(y) = 0$. From (1) it is clear that "additive poly-

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