ON HOMOGENEOUS RIEMANNIAN MANIFOLDS

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The object of this paper is to characterize homogeneous Riemannian manifolds (i.e. Riemannian manifolds whose group of isometries is transitive) in terms of the behavior of their curvature under parallel translation. We seek an extension of Cartan's characterization of symmetric Riemannian manifolds as those whose curvature is constant under parallel translation. We obtain a theorem of this nature under the assumption that the Riemannian manifold is complete and simply connected,—these being natural assumptions here since curvature parallel translates the same way in any locally isometric manifolds.

Our characterization is in terms of the existence of a certain function T which assigns to each x in any tangent space to M (M the Riemannian manifold) a skew-symmetric linear transformation T_x on that tangent space; the symmetric case is the special case where all T_x are zero. Thus we are characterizing Riemannian homogeneity of M (in case M is complete and simply connected) by a local condition which is to be satisfied at all points.

The conditions on T which we prove necessary and sufficient for Riemannian homogeneity are two: (A) a condition relating the covariant derivative of the curvature to the T_x, (B) a condition on the covariant derivative of the T_x. Condition (A) describes, in infinitesimal form, how curvature parallel translates, in terms of the T_x. In special cases this condition amounts to saying that the curvature varies as an exponential in the T_x but in general it gives a rule a little more complicated than an exponential rule. If all the T_x are zero, it reduces to the condition that the covariant derivative of the curvature be zero. Condition (B) (which is automatically satisfied if all T_x are zero) says, in infinitesimal form, that the isometries (to be constructed from the T_x or, if M is known to be homogeneous, which are given in advance) carry the T_x at one point into those at other points.

We give two proofs of our main theorem, which we believe to be of independent interest. The first proves M is homogeneous by constructing a Lie group G of which M is a quotient space. The interest of this proof lies in the way G is constructed, thru its Lie algebra, out of the T_x and the curvature of M. In fact this proof sets up a natural correspondence between groups G of which M is a Riemannian homogeneous space and T's which satisfy our conditions (A) and (B). This suggests the possibility of classifying the groups G of which M is a Riemannian homogeneous space thru these T's. It also suggests the possibility of classifying Riemannian homogeneous manifolds by properties of the

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