NATURALLY TOTALLY ORDERED COMPACT SEMIGROUPS

By Katsumi Numakura

1. **Introduction.** We recall that a *mob* is a Hausdorff space together with a continuous associative multiplication. Let A be a subset of a mob S. An element a of S is a *left zero for* A, if $aA = a$. A *left identity for* A is an idempotent u (i.e., $u^2 = u$) such that $uA \supset A$. Similarly, one defines *right zero* and *right identity*. By *zero* (*identity*) for A we shall mean both left and right zero (identity) for A. A zero (identity) for the mob is called briefly *zero* (*identity*). An element x of a mob S with zero, 0, is *nilpotent* (*algebraically nilpotent*), if $x^n \to 0$ ($x^n = 0$ for some $n \geq 1$).

An *(I)-mob* (terminology due to Mostert and Shields [5]) is a mob on a closed interval of the line such that one end point functions as a zero and the other as an identity for the mob. We denote by J_1 the (I)-mob [0, 1] under the usual multiplication and by J_2 the (I)-mob $[\frac{1}{2}, 1]$ with multiplication $x \circ y = \max(\frac{1}{2}, xy)$, where xy denotes the usual multiplication of real numbers. J_1 and J_2 are called *(J)-mobs*.

(I)-mobs were first studied by W. M. Faucett [3], who demonstrated that 1) if a compact connected mob S has just two idempotents and it is irreducibly connected between the idempotents (i.e., it contains no proper connected subset containing the idempotents), then S is an (I)-mob [3; Theorem 1], and 2) an (I)-mob with no other idempotents and no non-zero algebraic nilpotents must be topologically isomorphic to J_1 [3; Theorem 2]. The structure theorem of general (I)-mobs has been given by P. S. Mostert and A. L. Shields [5; Theorem B]. In particular, they proved that, if an (I)-mob with just two idempotents has at least one non-zero algebraic nilpotent, then it is (topologically) isomorphic to J_2 [5; Theorem 5.3.2].

The proofs of Faucett's theorems are based on the linear order introduced in the mob, using irreducible connectedness. He also remarked that this order relation coincides with the order relation defined as follows: for $a, b \in S$, $b \leq a$ if $bS \subset aS$ and $Sb \subset Sa$. Thus, under the order relation, S becomes a naturally totally ordered mob in the sense of A. H. Clifford [2].

In this note, we shall give some structure theorems of naturally totally ordered compact mobs. It is not assumed here, however, that the mob is commutative or connected. We say that, for $a, b \in S$, a divides b and write $a \geq b$, if either $a = b$ or there exists c in S such that $ca = b$. We write $a > b$ if $a \geq b$ but $b \ngeq a$. Notice that $a = b$ means strict equality. We call S *naturally totally ordered*, (n.t.o.), (terminology due to Clifford [2]), if the division relation is a total ordering of S: for every pair of elements a, b of S exactly one of the relations $a > b$, $a = b$, $a < b$ holds. We do not assume that the existence of $d \in S$ with $ad = b$ for $a > b$. Two cases arise, the connected case and the non-connected

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