A CENTRAL LIMIT THEOREM FOR m-DEPENDENT RANDOM VARIABLES

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It m is a non-negative integer, a sequence $\{X_v\}$ of random variables is *m-dependent* if $X_1, X_2, \cdots, X_s$ is independent of $X_t, X_{t+1}, \cdots$ provided $t - s > m$; a similar definition applies to finite sequences of random variables. A central limit theorem for m-dependent random variables was presented in [2]. Related and improved results were proved by Diananda, the best result appearing in [1]. The proof in [1] assumes not only the natural generalization of the Lindeberg conditions but also uniformly bounded variances and another condition on the rate of growth of the variances of the partial sums. The theorem we prove below has conditions which, for the case where $m = 0$ and finite second moments exist, reduce to the Lindeberg conditions. Actually we assume no moments so that when $m = 0$, our conditions reduce to those of Khintchine.

We shall be concerned with an infinite sequence of finite sequences of random variables, $\{X_{nv}\}$, $v = 1, \cdots, p(n)$, $n = 1, \cdots$. To avoid excessive subscripts dependence on n will usually be suppressed. Thus $\sum_{v=1}^{p} P[| X_v | > \epsilon]$ is written for $\sum_{v=1}^{p(n)} P[| X_{nv} | > \epsilon]$. If $\tau > 0$, X^τ stands for the random variable X truncated at τ. If M is a set, X^M stands for the random variable agreeing with X on M and equalling zero elsewhere. We write σ_{vw}^τ for $\int ((X_v^\tau - \int X_v^\tau)(X_w^\tau - \int X_w^\tau))$ and α_v^τ for $\int X_v^\tau$, where here and elsewhere integration is with respect to the underlying probability measure. Note that σ_{vw}^τ, α_v^τ depend on n since the defining expressions contain suppressed subscripts n. All limits are taken as n approaches infinity.

THEOREM. *Let $\{p(n)\}$ be a non-decreasing sequence of positive integers approaching infinity, m a non-negative integer, and τ some positive real number. If* (1) *$\{X_{nv}\}$, $v = 1, \cdots, p(n)$, $n = 1, \cdots$, is a sequence of sequences of m-dependent random variables,* (2) *$\sum_{v=1}^{p} \alpha_v^\tau \to \alpha$,* (3) *$\sum_{v=1}^{p} \sum_{w=1}^{p} \sigma_{vw}^\tau \to \sigma^2$,* (4) *$\sum_{v=1}^{p} P[| X_v | > \epsilon] \to 0$ for all $\epsilon > 0$,* (5) *$\sum_{v=1}^{p} \sigma_{vv}^\tau = O(1)$, it follows that* (*) *$\sum_{v=1}^{p} X_v$ converges in distribution to the normal with mean α, variance σ^2.*

LEMMA 1. *If $\{X_{nv}\}$ satisfies* (1)–(5) *of the theorem with $m = 1$ and $\alpha = 0$ and* (6) *$\sum_{v=1}^{p} | \alpha_v^\tau | \to 0$, then* (*) *holds.*

Proof. Let $\{X_{nv}\}$ satisfy the hypotheses of the lemma. It follows from (4) that there exists a non-decreasing sequence $\{k'(n)\}$ of positive numbers such that $k'(n) \to \infty$, $k'(n) = o(n)$ and $\sum_{v=1}^{p} P[| X_v | > 1/k'] \to 0$. Letting $k(n)$ be the largest even integer in $\sqrt{k'(n)}$ it follows that $k(n)$ is a non-decreasing sequence of positive even integers, $k(n) \to \infty$, $k(n) = o(n)$ and

Received August 21, 1957.