INTEGRAL INEQUALITIES OF THE WIRTINGER TYPE

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0. **Introduction.** This paper is concerned with generalizations of integral inequalities of the following type. (For the first three results quoted, see [4; 184–7]; the last two were proved recently in [3]. As in [4; 173], whenever u and u' occur in a theorem, u is an integral, namely the integral of u'.)

Theorem I. *If $u' \varepsilon L_2$, and $u(0) = 0$, then*

$$\int_0^{\pi/2} u'^2\,dx > \int_0^{\pi/2} u^2\,dx, \quad \textit{unless} \quad u = A \sin x.$$

Theorem II. *If $u' \varepsilon L_2$, and $u(\pm\pi/2) = 0$, then*

$$\int_{-\pi/2}^{\pi/2} u'^2\,dx > \int_{-\pi/2}^{\pi/2} u^2\,dx, \quad \textit{unless} \quad u = A \cos x.$$

Theorem III. *If $u' \varepsilon L_2$, $u(-\pi) = u(\pi)$, and $\int_{-\pi}^{\pi} u\,dx = 0$, then*

$$\int_{-\pi}^{\pi} u'^2\,dx > \int_{-\pi}^{\pi} u^2\,dx, \quad \textit{unless} \quad u = A \cos x + B \sin x.$$

Theorem IV. *If $u'' \varepsilon L_2$, and $u(0) = u(\pi) = 0$, then*

$$\int_0^{\pi} u''^2\,dx > \int_0^{\pi} u^2\,dx, \quad \textit{unless} \quad u = A \sin x.$$

Theorem V. *If $u'' \varepsilon L_2$, $u'(0) = u'(\pi) = 0$, and $\int_0^{\pi} u\,dx = 0$, then*

$$\int_0^{\pi} u''^2\,dx > \int_0^{\pi} u^2\,dx, \quad \textit{unless} \quad u = A \cos x.$$

Theorem III is called Wirtinger's inequality. It, together with Theorem V, differs from the remaining theorems in that the admissible functions are restricted not only by boundary conditions, but also by a condition of the type $\int u\,dx = 0$. As a matter of fact, all these inequalities could be stated so as to involve a condition of this type. This condition is seen to be an orthogonality condition when viewed in a more general setting. We shall be considering integral inequalities of the forms

$$\int_a^b u'^2\,dx \geq \int_a^b pu^2\,dx, \tag{0.1}$$

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