INVERSIONS OF GENERALIZED LAMBERT TRANSFORMS

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Introduction. The Laplace transform may be viewed as a continuous analogue of power series. The continuous analogue of the Lambert series

$$F(z) = \sum_{k=1}^{\infty} \varphi_k \frac{z^k}{1 - z^k}$$

is called the Lambert transform

$$F(x) = \int_{0+}^{\infty} \frac{1}{e^{xt} - 1} \varphi(t) \, dt. \tag{1}$$

The transform (1) has been inverted in the case where $\int_0^\infty | \varphi(t) |/t \, dt < \infty$ and $\varphi(t) = O(t^\delta)$ as $t \to 0^+$ for some $\delta > 0$. (See [5; 171].)

In this paper we will generalize this result in several directions. First we will consider transforms with kernels of the form $K(xt) = \sum_{k=1}^\infty a_k e^{-kxt}$ for a fairly general class of sequences $\{a_k\}$. (When $a_k = 1$ for $k = 1, 2, \cdots$, we have $K(xt) = 1/(e^{xt} - 1)$—the kernel of the Lambert transform (1).) Second, we will work with transforms as Stieltjes integrals

$$F(x) = \int_{0+}^{\infty} K(xt) \, d\alpha(t) = \int_{0+}^{\infty} \sum_{k=1}^{\infty} a_k e^{-kxt} \, d\alpha(t), \tag{2}$$

a special case of which is

$$F(x) = \int_{0+}^{\infty} K(xt)\varphi(t) \, dt = \int_{0+}^{\infty} \sum_{k=1}^{\infty} a_k e^{-kxt} \varphi(t) \, dt. \tag{3}$$

We will also endeavor to show that the hypotheses under which our inversion is obtained are as weak as possible. In particular, to invert (1) we need only assume that the integral converges for some $x > 0$ and that $\int_0^1 | \varphi(t) \log t |/t \, dt < \infty$ (Theorem 7.7).

By a *Generalized Lambert transform* then, we mean the transform (2) or (3). For brevity we will refer to these as *L-transforms*. We will henceforth assume that $\alpha(t)$ is a normalized function of bounded variation on $0 \leq t \leq R$ for every $R > 0$, that $\varphi(t) \in L$ on $0 \leq t \leq R$ for every $R > 0$, and that in (2) or (3)

$$\int_{0+}^{\infty} = \lim_{\substack{\epsilon \to 0 \\ R \to \infty}} \int_{\epsilon}^{R} .$$

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